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Promoting the development of procedural flexibility

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Overview of talk

- Who am I?
- Let's talk about algebra
- Let's talk about procedural flexibility
- Let's talk about comparison
- Three relatively simple ways that teachers can take advantage of comparison to improve students' procedural flexibility
- What can this look like in practice?

Who am I?

- Taught middle and high school mathematics for 6 years
  - Favorite course to teach was Algebra I
  - Intrigued and challenged by students’ difficulties in learning algebra
- PhD in Education and Psychology from University of Michigan
  - Focused on students’ learning of algebra

Who am I?

- Assistant Professor at Graduate School of Education at Harvard for the past 4 years
  - Teach methods courses for secondary preservice math teachers
  - Teach graduate courses in students' learning of school subjects such as math
- Served on the faculty at Michigan State University for 5 years

What am I interested in?

- Students' strategies for solving problems in mathematics, but especially in algebra
- What strategies do students use to solve problems?
- How do students know what strategies to use on unfamiliar problems?

Why focus on strategies?

- Addresses many noted difficulties for students in algebra and in mathematics more generally
- Only apply procedures by rote
- Strategies easily forgotten or applied incorrectly
- Don’t understand the strategies that they are using
- Difficulty in approaching unfamiliar problems

Knowledge of strategies

Superficial knowledge: Knowledge associated with rote learning, inflexibility, reproduction, and trial/error

does not understand "**what to do and why**"

does understand "**what to do and why**"

Deep knowledge (procedural flexibility): Knowledge associated with comprehension, abstraction, flexibility, and critical judgment

Procedural flexibility

- Strategic, rather than rote, use of procedures
- Knowledge of multiple strategies for solving a type of problem
- Ability to choose the most appropriate strategy for a given problem or situation

Let’s talk about algebra

- Procedural flexibility has particular relevance for algebra
- What is algebra? What is it hard for students? Why is it important for students?

What is algebra, to me?

- Algebra is mostly about:
  - functions?
  - expressions and equations and gaining fluency with symbols?
  - exploring and representing relationships between varying quantities?
  - algebraic reasoning
- Certainly all of these are important

Why is algebra *hard*?

1. Lack of clarity among teachers and researchers about ***what algebra is***

- Hard to design curriculum and effectively teach students if we can't agree on what algebra is

(*More on this in a moment...*)

Why is algebra *hard*?

2. For parents, consensus about what algebra is:

- Algebra is about doing ***mysterious things with the last three letters*** of the alphabet?!
  - Most adults don't use algebra in their daily lives?!
  - Why should students learn algebra at all?!
  - Why don't we focus on other math topics that are more important and relevant to real life?!
- Difficult for parents to be supportive of our efforts in algebra with this view of the topic

Why is algebra *hard*?

3. Algebra represents a leap in abstraction for students, and this is hard!

- Moving from the tools and ways of reasoning used in arithmetic, which often involve counting or working with concrete objects, to the more abstract reasoning and generalization processes involved in algebra, is a long and hard process for many students

Why is algebra so *important*?

1. Completing an algebra course is linked to success in later math courses, college graduation rates, ability to get certain kinds of jobs, etc.

- Algebra is considered a critical ***gatekeeper course***, in that taking and passing algebra opens up educational and career doors that would otherwise be closed

Why is algebra so *important*?

2. Algebra, and the reasoning that is a part of algebra, is a ***core component of the discipline of mathematics***

- One reason that students take math in school is to get exposure to what mathematics is all about, and algebra and algebraic reasoning play an extremely important role in mathematics

Why is algebra so *important*?

3. Some also make the argument that students should take algebra because ***it is useful in daily life***

- We are constantly using algebra in our personal and professional lives, even when we don't know we are using algebra

What *is* algebra?

- At present, two main ways of thinking about algebra in the US mathematics curriculum
- A ***function-based*** view
- An ***equation-based*** view

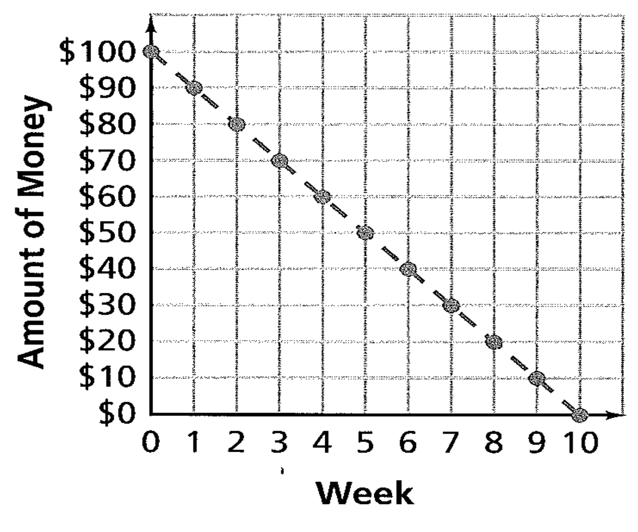
Function-based view of algebra

- Algebra is primarily about relationships between quantities that vary, how to represent these relationships, and how to use representations to ***analyze, generalize, and make predictions*** about these relationships

Function-based example

- Sarah decided to spend her savings to buy music for her iPod. She has $100 now and spends $10 each week. How much money does she have in her savings account?

Week	$
0	100
1	90
2	80
3	70



Equation-based view of algebra

- Algebra is primarily about modeling relationships with symbols and then ***using symbols*** to determine features of these relationships

Equation-based examples

1) Solve for *y* when x = 4

$$y = 3x - 0.5$$

2) What is the value of $(15x^4)^0$?

3) Combine: $\frac{5x}{6} - \frac{3x}{12} - \frac{x}{3}$

Connecting both views of algebra

- Algebra is about both of these views, but ***we've found it difficult to connect function- and equation-based views in our instruction***
  - Also, these two views are separated temporally - function-based views are present in many middle school curricula but equation-based views are present in most high school curricula
- Even when both views exist simultaneously in an algebra course, ***they are not well-integrated***

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ICATIONS CONNECTIONS

Another way to think about algebra

- Instead of “What is algebra?”, instead ask
- “What does it mean to ***understand*** algebra?”

Understanding in algebra

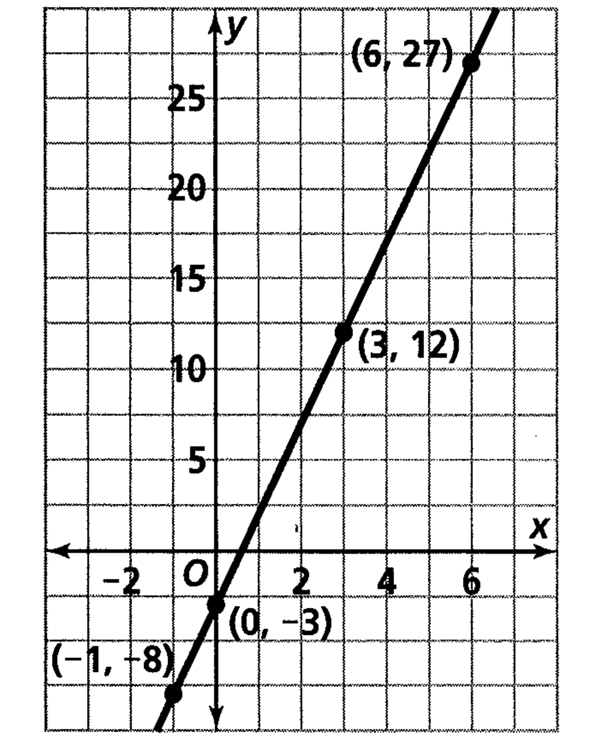
- Representations: Symbols, graphs, tables
  - Representations are useful to analyzing relationships between quantities
- Understanding in algebra is:
  - Ability to move fluently ***between*** multiple representations
  - Ability to operate fluently ***within*** each representation

Representations

Tabular

x	y
0	-3
1	2
2	7
3	12

Graphical



Symbolic

y = 5*x* - 3

Between representations

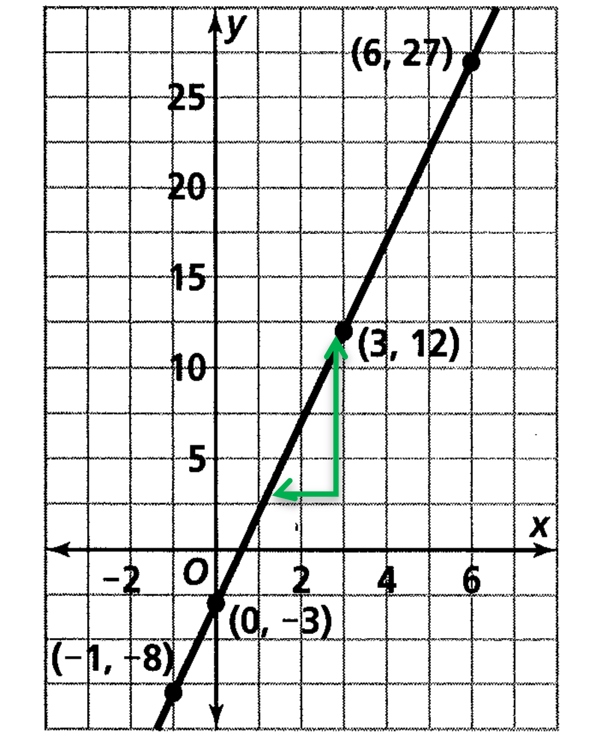
- Analyze situations using graphs, tables, and symbols
- ***Make connections*** between graphs, tables, and symbols

Example of “between”: slope

Tabular

x	y
0	-3
1	2
2	7
3	12

Graphical



Symbolic

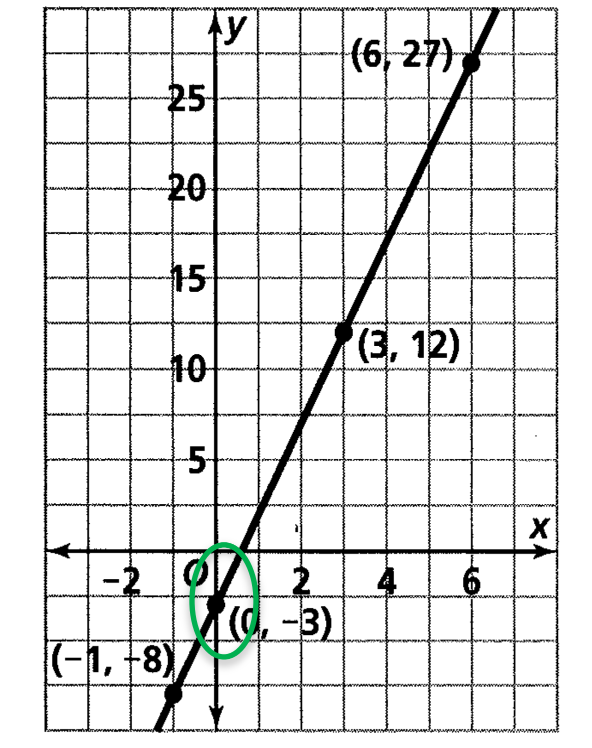
$y = 5x - 3$

Example of "between": y intercept

Tabular

x	*y*
0	-3
1	2
2	7
3	12

Graphical



Symbolic

$y = 5x - 3$

Within representation

- Know concepts and skills for working with a single representation
- But also ***know multiple strategies*** in a representation, including the ability to ***select the most appropriate strategies*** in that representation for a given problem
- ***Flexibility*** within a given representation

‘Within’ example using tables

Create a table for the following equation:

$$y = \frac{1}{3}x + 4$$

Standard Way:

x	y
0	4
1	4 1/3
2	4 2/3
3	5

Short Way:

x	y
0	4
3	5
6	6
9	7

‘Within’ example #1 using symbols

- Time to do some math! Solve this equation for *x*:

$$3(x + 1) = 15$$

- Can you solve this same equation again, but in a different (better?) way?
- Now solve this equation for *x*:

$$3(x + 1) = 14$$

Problem 1 Strategy 1

$3(x + 1) = 15$

$3x + 3 = 15$

$3x = 12$

$x = 4$

Problem 1 Strategy 2

$3(x + 1) = 15$

$x + 1 = 5$

$x = 4$

Problem 2 Strategy 1

$3(x + 1) = 14$

$3x + 3 = 14$

$3x = 11$

$x = 11/3$

Problem 2 Strategy 2

$3(x + 1) = 14$

$x + 1 = 14/3$

$x = 11/3$

So what?

- We want students to *know multiple strategies* in a representation, including the ability to *select the most appropriate strategies* in that representation for a given problem
- Choosing most appropriate strategy
  - reduces error
  - allows for a quicker solution
- Flexibility is linked to conceptual knowledge
  - ideas about equivalence and variables
- Flexibility is linked to transfer performance

‘Within’ example #2 using symbols

- Solve the following problems for x:

Problem 1: $$\frac{x}{14} = \frac{16}{7}$$

Problem 2: $$\frac{x}{14} = \frac{16}{8}$$

'Within' example #2

- Strategy #1: Cross multiplication strategy

$$\frac{x}{14} = \frac{16}{7}$$

$7x$ = ???

- Strategy #2: Equivalent fractions strategy

$$\frac{x}{14} = \frac{16}{7}$$

7 times 2 is 14,
so 16 times 2 is 32

‘Within’ example #2

- Strategy #1: Cross multiplication strategy

$$\frac{x}{14} = \frac{16}{8}$$

$8x$ = ???

- Strategy #2: Unit rate strategy

$$\frac{x}{14} = \frac{16}{8}$$

8 times 2 is 16,
so 14 times 2 is 28

All of these ‘within’ examples...

- Involve knowing multiple strategies to approach a class of problems and selecting the most appropriate strategy for a particular problem
- Move students beyond rote application of a single procedure to more flexible use of multiple strategies
- Procedural flexibility

Developing flexibility

- How can teachers help students to develop procedural flexibility?
- Ideally, interventions would be
  - easy to implement
  - have big impact on students
- What kinds of small changes in teachers' practices appear to have a significant impact on students' flexibility in algebra?
- I will discuss three - all draw on the benefits of comparison

Comparison

- Is a fundamental learning mechanism
- Strong empirical support for benefits of comparison from cognitive science
- Helps focus our attention on critical features of what we are trying to learn
- For example...

Buying a camera at Best Buy

- How do I decide which camera to buy, and how does comparison help?
- Let's say I start with price and pick two cameras at the same price that both look OK

Canon SD-30, $299.99

Canon SD-1000, $299.99

Look at camera #1 (SD-30)

SD-30 Information

Canon SD-30, $299.99

Look at camera #2 (SD-1000)

SD-1000 Information

Canon SD-1000, $299.99

Limits of sequential viewing

- Hard to remember information about the SD-30 when looking at the SD-1000 and vice versa
- Hard to know which features the cameras are similar on and which they are different on
- As a result, I am just as likely to be influenced by *superficial* features (shape and color) than by ***substantive*** features

What if I compare?

Comparing these two cameras

Canon SD-30, $299.99

Canon SD-1000, $299.99

Comparison

- Makes it easier for me to see the features where these cameras are the same and different
- In the case of mathematics learning...

Comparison helps students...

- See similarities and differences between problems, representations, and strategies
- See how and why a strategy works for a particular problem
- See how and why a representation is particularly useful for answering certain kinds of questions
- See why some strategies or representations are better than others for certain kinds of problems

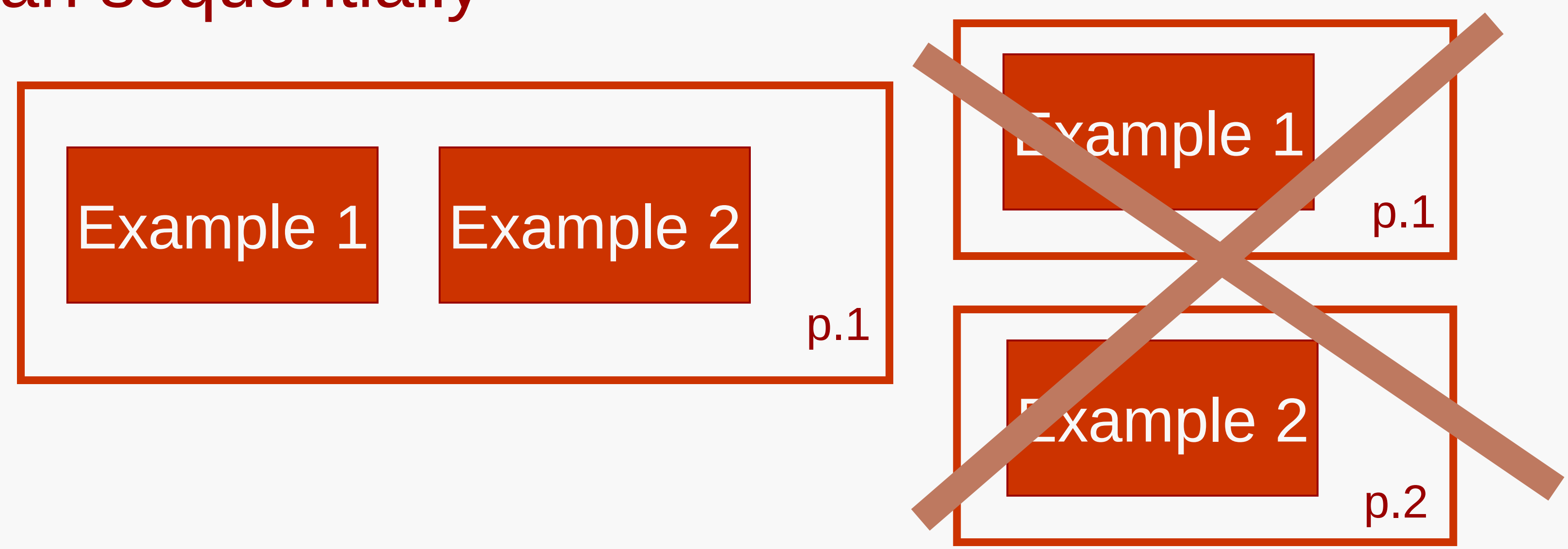
3 little things that really help

- Three small, interrelated interventions to help students become more successful in algebra:

1. Alter instructional presentation of problems and strategies so that comparison can occur
2. Engage students in conversations that are focused on comparison
3. Provide opportunities for students to generate multiple approaches so that comparison can occur

1. Instructional presentation

- Instructional materials and presentation should provide opportunities for students to see multiple problems and strategies at the same time, rather than sequentially



Can take several formats

- Display the same problem solved two different ways, side by side (compare strategies)
- Display two different problems solved in the same way, side by side (compare problem types)
- Both of these ways of presenting problems are better for student learning than sequential presentation

Compare strategies example

$$3(x + 1) = 15$$
$$3x + 3 = 15$$
$$3x = 12$$
$$x = 4$$

- Students see one worked example

Compare strategies example

$$3(x + 1) = 15$$
$$x + 1 = 5$$
$$x = 4$$

- Followed by another one on a different page (but same problem)
- Hard to compare these two strategies when presented in this way
  - Think back to the camera example

Compare strategies example

Strategy 1	Strategy 2
$3(x + 1) = 15$	$3(x + 1) = 15$
$3x + 3 = 15$	$x + 1 = 5$
$3x = 12$	$x = 4$
$x = 4$	

- But when presented together, we can compare much more easily
- Can see that these problems are same
- Can make sense of the innovative use of the divide step
  - More likely to adopt this strategy

Important caution

- The compared examples need to differ on some important dimensions but not be completely different
- How the examples differ needs to be important and relevant to the problem solving process

2. Comparison conversations

- No evidence that altering instructional presentation by itself is sufficient
- Needs to be combined with providing opportunities for students to compare and evaluate different strategies that they are learning
- Two types of conversations
  - similarities and differences in strategies
  - evaluation of strategies

Similarities and differences

- How are these strategies similar?
- How are they different?
- How are these strategies related to other strategies that we have used before?

Evaluation of strategies

- Which strategy is better for this problem and why?
- Why is this strategy the most effective (or most efficient, or most elegant, or the best) to use for this problem?
- On what kinds of problems is this strategy most effective (or most elegant, or the best)?

Important cautions

- Avoid "serial sharing" conversations
  - This does not provide opportunities for comparison
- Teachers need to provide visual and gestural cues to aid comparison
  - Evidence from TIMSS that teachers in high performing counties do more of this than US teachers

3. Multiple approaches

- Instruction should provide opportunities for students to generate multiple ways to solve the same problem

Several different formats

- Students are given one solution to a problem and are asked to generate another, different solution
- Students are given a problem and asked to solve it in two different ways
- Whole class discussion where multiple different solutions are generated
  - “Can anyone think of another, different way to approach this problem that we haven’t seen yet?”

Link to comparison

- The act of producing solution methods that are different requires some comparison
- Presence of multiple solution strategies to the same problem links to other two recommendations
  - Instructional presentation which shows two or more strategies side by side
  - Evaluative conversations about the multiple solution strategies

Important caution

- Cannot be busy-work!
- Instructional emphasis is not on the mere generation of multiple methods but on the subsequent conversations and reflections that this task can lead to
- Differences in generated solutions methods may be trivial or not relevant
  - Another reason why this task needs to be combined with some sort of evaluative discussion

Bottom line

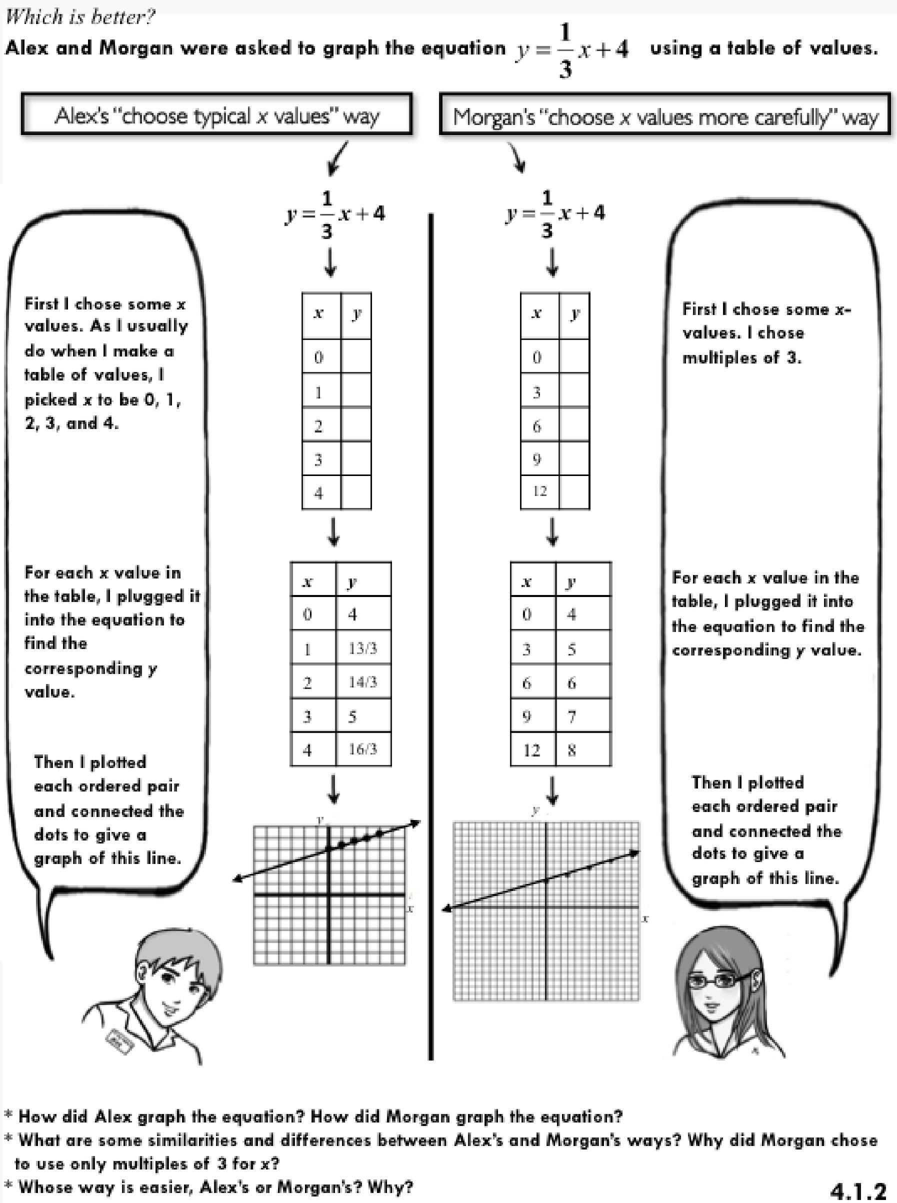
- Three related but relatively small instructional interventions that have a significant impact on student learning
  - Show multiple examples side by side
  - Discuss and evaluate multiple strategies
  - Have students generate multiple ways

In sum...

- It pays to compare!
- Comparison is linked to gains in procedural flexibility in algebra
- Three related but relatively small instructional interventions that have a significant impact on student learning
  – Show multiple examples side by side
  – Discuss and evaluate multiple strategies
  – Have students generate multiple ways

What might this look like?

- We have developed supplemental curriculum materials (“worked example pairs”) that draw upon these interventions and are testing them in algebra classes
- We are in the midst of a large randomized controlled trial in Massachusetts evaluating the impact of these materials on students’ learning of algebra



Which is better?

Alex and Morgan were asked to graph the equation $y = \frac{1}{3}x + 4$ using a table of values.

Alex's "choose typical x values" way

$y = \frac{1}{3}x + 4$

x	y
0	
1	
2	
3	
4	

x	y
0	4
1	13/3
2	14/3
3	5
4	16/3

First I chose some x values. As I usually do when I make a table of values, I picked x to be 0, 1, 2, 3, and 4.

For each x value in the table, I plugged it into the equation to find the corresponding y value.

Then I plotted each ordered pair and connected the dots to give a graph of this line.

Morgan's "choose x values more carefully" way

$y = \frac{1}{3}x + 4$

x	y
0	
3	
6	
9	
12	

x	y
0	4
3	5
6	6
9	7
12	8

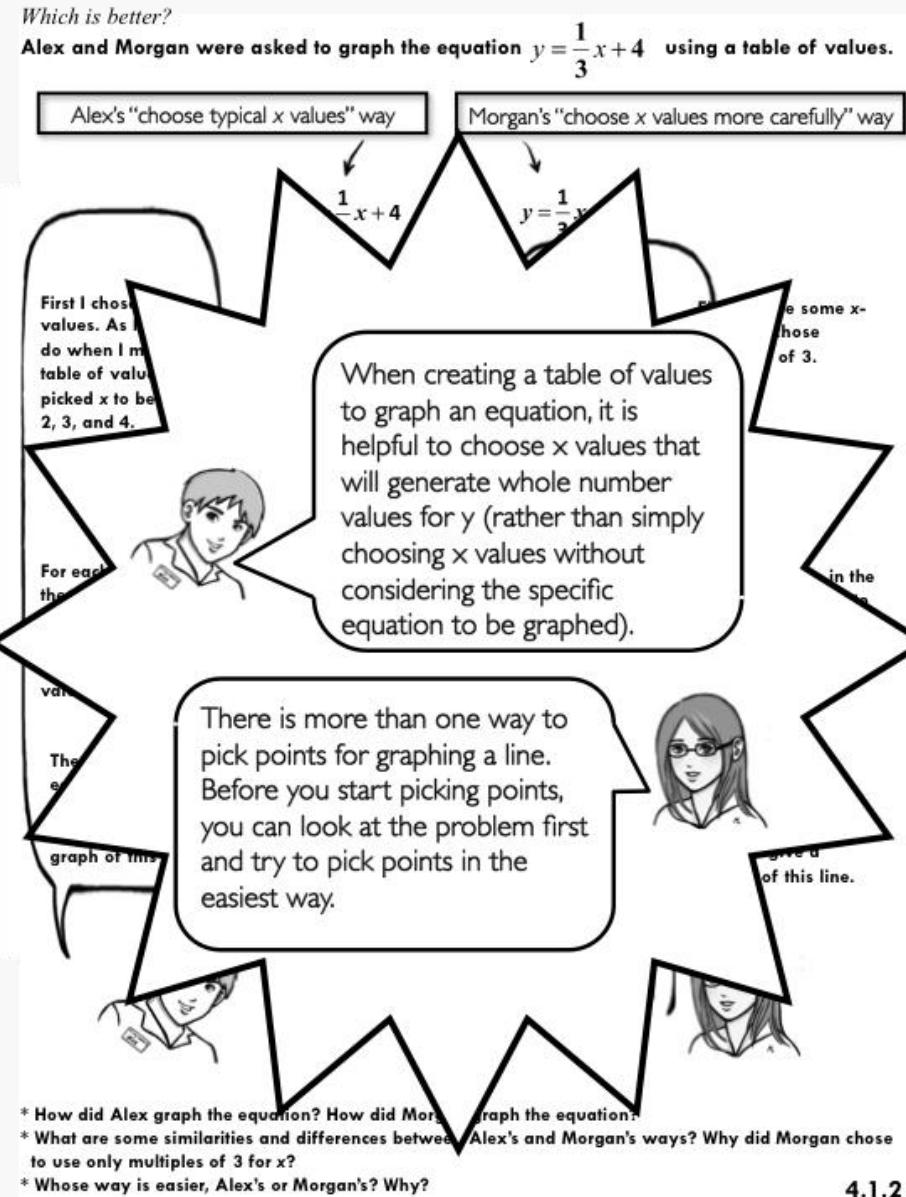
First I chose some x-values. I chose multiples of 3.

For each x value in the table, I plugged it into the equation to find the corresponding y value.

Then I plotted each ordered pair and connected the dots to give a graph of this line.

* How did Alex graph the equation? How did Morgan graph the equation?
* What are some similarities and differences between Alex's and Morgan's ways? Why did Morgan chose to use only multiples of 3 for x?
* Whose way is easier, Alex's or Morgan's? Why?

4.1.2



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* Whose way is easier, Alex's or Morgan's? Why?

4.1.2

Let’s go to the video!

- Context
  - A teacher using very early version of our materials
  - Private school, small classes, 9th grade Algebra I
- As you watch video, look for the ways that the teacher makes use of the three instructional recommendations
  - Show multiple examples side by side
  - Discuss and evaluate multiple strategies
  - Have students generate multiple ways

Current study

- One week professional development institute
- Approximately 150 worked example pairs that can be used for all topics in the Algebra I curriculum
- About 40 teachers using these materials, videotaping themselves, giving students tests
  - An equal number serving as the "comparison group"
- Preliminary results in a year or so
- I can't share curriculum materials yet!

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Questions? Comments? Thanks!

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