MODERATOR: Good morning. We'd like to get started. Welcome to this session on Promoting the Development of Procedural Flexibility with our presenter, Dr. Jon Star. Let me tell you a little bit about Dr. Star. He is an Educational Psychologist who studies children's learning of mathematics in middle and high school, particularly algebra. Dr. Star's current research explores the development of flexibility in mathematical problem solving, with flexibility defined as knowledge of multiple strategies for solving mathematics problems and the ability to adaptively choose among known strategies on a particular problem.

Dr. Star also investigates instructional and curricular interventions that may promote the development of mathematical understanding. His most recent work is supported by grants from the Institute of Educational Sciences at the U.S. Department of Education and from the National Science Foundation. In addition, Dr. Star is interested in the pre-service preparation of middle and secondary mathematics teachers. Prior to his graduate studies, Dr. Star spent six years teaching middle and high school mathematics. Let's welcome Dr. Star.

DR. STAR: Thanks, everyone. Let's make sure the mic is not going to be too loud here. Well, thanks for staying until the end of a long week for this session. I know that it's the end. But I'm really pleased to be here with you and to talk a little about procedural flexibility. This is a relatively small group, so I do have some slides and some remarks I'd like to make. But I hope that this can be as interactive as possible. Feel free to interrupt me to lead us into tangents for discussion. There's plenty of time in a two hour session for a group this size.

So before I begin, I just want to acknowledge, as you heard in the introduction, that the work I'm going to be talking about is supported by the National Science Foundation and the U.S. Department of Education. And I have some collaborators in this work. Can you see through me? Yeah? I have some collaborators, one at Vanderbilt and one at Temple who have been really instrumental in this work.

My plan for my time here is, I'd like to tell you a little more about me to help situate the things I'll be talking about. And I do want to talk about algebra, because the work that I do is of particular relevance to algebra. I'll talk about procedural flexibility. I'll talk about comparison. I'll talk about three relatively simple ways that teachers can take advantage of comparison to promote the development of procedural flexibility. And then, I have a video that I'd like to show you and talk about that shows you what this might look like in practice.

So before I jump into these, it actually would help me to learn a little about who you are to figure out who I'm speaking to. So how many of you are teachers, full time, that's what you do, teachers? And of those with the hands raised, what level do you teach? High school? Okay.

WOMAN: High school special ed.

WOMAN: High school.

DR. STAR: High school?

WOMAN: Middle school, section one.

WOMAN: We're elementary.

DR. STAR: Do you teach upper, lower, or all levels of elementary?

WOMAN: Fourth grade and third grade.

DR. STAR: Okay, good. So those who didn't raise their hands, are you coaches? How many of you are coaches? Okay. And what level do you coach?

WOMAN: (inaudible) middle school . . .

DR. STAR: And then, the others? Administrators?

WOMAN: Consultant.

DR. STAR: Consultants?

WOMAN: (inaudible)

WOMAN: (inaudible)

DR. STAR: And those, the consulting, so are you at all levels or . . .

WOMAN: K-12.

DR. STAR: K-12? Okay, good. Well, that helps me a lot. As I'll talk about, I'll be speaking about the work that I do, I would say it's centrally located in the grade band of about grades five to ten. That's sort of where I see algebra happening these days. But I think it does have relevance to before that and after that. But the examples that I'll use are in that kind of five to ten grade band.

So who am I? As you heard, I'm a former middle school and high school math teacher. When I was a teacher, I was really interested in the Algebra 1 course. In particular, as no surprise to all of you, a lot of students have trouble with algebra, and I was really intrigued and challenged by these difficulties. And that lead me to think about going back to graduate school, where I could spend all of my time thinking about these challenges.

So I got a Ph.D. from the University of Michigan, and I really focused on students' learning of algebra. I've been at Harvard for four years now. We have a very small teacher education program at Harvard, but I'm the methods instructor for the middle and secondary school students in that program, and I also teach graduate courses about learning, generally in about learning of math. And prior to that, I was on the faculty at

Michigan State for five years. So for those of you who consider yourselves Midwesterners, I also have that in my background a bit too.

My interest is really in student strategies for solving problems in math, especially in algebra. And by that, what I really am interested in is, what strategies do students use when they solve problems that we give them and how do they know what strategies to use on unfamiliar problems? This seems to be the crux of lots of students' difficulties. They're given a problem, they haven't seen that problem before, they have a hard time figuring out what to do to approach that problem, and that's what I'm interested in.

So why this focus on strategies? Why do I think this is a big deal that I should devote my attention to? Well, I think it addresses many of the noted difficulties for students in algebra and mathematics that we're all very aware of.

We hear all the time, we experience this challenge that students only know how to apply procedures by rote. We hear that phrase a lot. We know that the strategies that we teach, they're easily forgotten or they're applied incorrectly, and they don't understand what they're doing. When we give them problems they've never seen before, they have difficulty approaching them. So, for me, all of these difficulties are about strategies, and that's why I've decided to really think carefully about what strategies students are using and how they know what strategies to use?

If you think about students' knowledge of strategies, I like to consider it as a continuum. There has to be a number line in the math talk, right? So on one side of the continuum, you've got maybe a student who does not understand this phrase that sometimes we use in the research world, what to do and why. They don't understand what to do and why. And on the other side, the student does understand what to do and why.

And just to introduce a little more terminology, I consider this left side of the does not understand knowledge that students have, that's superficial. It's knowledge that's associated with rote learning, inflexibility, reproduction, trial and error, and it's very fragile. And this is, unfortunately, where a lot of our students are.

I'm interested in figuring out how to get students on the other end of the continuum, which is what I call deep knowledge of strategies. And this is where procedural flexibility, which I'll talk about a lot more, comes in. And this is knowledge associated with comprehension, abstraction, flexibility, and critical judgment.

This terminology, for me, tries to get out of overuse of the word understand. I find that we use that word a lot without really talking about what that means or what that looks like. And so, I talk about this other outcome, about deep knowledge, about procedural flexibility, which is specifically applied to the strategies students know. And I think about how to get students toward this outcome.

So let me talk a little about procedural flexibility then. It's all about strategic, rather than rote, use of procedures. The idea here is that I want students to know multiple strategies for solving certain kinds of problems, and I want them to have the ability to choose the most appropriate strategy for a given problem or situation.

So in a few minutes, I'm going to give you some math problems to solve, and we'll talk some math, and I'll really try to make this concrete about exactly what I'm talking about.

But since this work is really situated for me in algebra broadly defined, I wanted to take a few minutes to talk about algebra, just so that when, because I found in my dealings with lots of people, algebra is a big deal these days, as you know. And what one person means by algebra is not necessarily what another person means by algebra. And so, I felt like it was important for me to talk about what I mean by algebra and what I'm trying to address in the domain of algebra. So let's talk about algebra for a moment.

And perhaps later, when we open this up for discussion, I'd be interested to know what's going on in Pennsylvania with algebra because I don't know much, but I see there was a talk earlier this week about this.

So as I mentioned, procedural flexibility, I think, has particular relevance for algebra. And I'm going to talk about what I think algebra is, why is it hard for students, why is it important for students, just to situate my remarks.

So when we think about what algebra is, it's different things to different people. For some people, algebra is really just about functions and the study of functional relationships. For other people, algebra is about expressions and equations, and getting fluency with symbols. Other people would say, well, no, it's actually about exploring and representing relationships between varying quantities, that's what algebra is all about. And other people would start to use phrases like algebraic reasoning to talk about what algebra is.

And so, at least in the research community, there's lots of battles about what algebra is and what we're trying to do in algebra. The truth is that algebra is all of these things. All of them are important. But, in fact, that is actually one of the reasons why I think algebra is hard for students, is that we don't all agree about what algebra is. It's hard to design curriculum and effectively teach students if we can't agree on what we're after. And that, I think, is a big problem with algebra, and that does impact our students. It makes it difficult for them. I'll say more about this in a moment.

Now interestingly, though among teachers and researchers there isn't necessarily agreement over what algebra is, there absolutely is quite strong consensus among parents what algebra is. And for them, it's about doing mysterious things with the last three letters of the alphabet. That's what algebra is for parents.

So when you talk to many adults, they don't really see that algebra is useful in their daily lives. They don't do mysterious things with Xs, Ys, and Zs in their daily lives. So why should their students, their children, be learning algebra at all? Why don't we focus on other math topics that are more important and relevant to real life? So this is a common sentiment that I hear a lot from parents, reflecting a certain view of algebra.

And certainly, it's difficult for parents to be supportive of our efforts in the algebra classroom if they have this view of the topic, and I find that's especially true in the current situation in many states where you're not just doing algebra in the course called Algebra 1. You're doing algebra in lots of other courses across a huge grade band. And so, often we get this reaction.

So another reason that I think algebra is hard, that I know is no surprise to most of you, is that it's a huge leap in abstraction for students. And it is just hard, inherently hard. In algebra, students are moving from the tools and ways of reasoning that they used in arithmetic, which typically involve counting or working with concrete objects, to the more abstract reasoning and generalization processes that are involved in algebra. And this is a long and hard transitional process for students.

Despite these challenges, despite the fact that it's hard, that parents often don't agree with our conception of algebra, algebra is very important, as we all know, and there's a lot of good data out there that show that taking an algebra course is linked to success in later math courses, college graduation rates, ability to get certain kinds of jobs. You name the positive outcome, it is linked to success in algebra. There's good data on that.

And I like to raise this just because many teachers that I work with are certainly aware that algebra is a big deal in their state, but sometimes there's questions about why? Why did algebra become such this big thing that we're talking about? And it's because there is good data on that, that kids who complete an Algebra 2 course are, I think the data was something like two-thirds more likely to graduate from college, if they can make it doing an Algebra 2 course.

And there's similar data about Algebra 1. If you complete an Algebra 1 course, your income is much higher than if you don't complete an Algebra 1 course. It's viewed as a critical gatekeeper course. It opens up educational and career opportunities that otherwise would be closed.

Now I have another reason I think algebra is very important, and I think this is a little more subtle point to make. And I think algebra is important because algebra and the reasoning that's part of algebra is a core part of what mathematics is all about. It plays a critical and core role in the discipline of mathematics.

One reason that students take math in school is to get exposure to what mathematics is all about, and algebraic reasoning plays an important role in the discipline of mathematics. So this, for me, is a disciplinary argument for why algebra is important. I could talk a lot more about that. I think that's, if you have questions or that's a point of conversation, we can come back to that.

A third reason is that some people make the argument that students should take algebra because it's useful in daily life. Now as I mentioned, many parents don't see how this is the case. But many people do feel strongly that algebra is useful, and they would argue that we're constantly using algebra in our personal and professional lives, even when we don't know we're using algebra. It's a little bit of a hard sell to tell someone they should take something because they're using it, even when they don't know they're using it. But for some people, this is a reason why we teach algebra.

So what is algebra? At present, I would argue there's two main ways that we're thinking of algebra in the U.S. math curriculum. There's one that I call a function-based view, and there's one that I call an equation-based view, and let me speak to these two. And again, this is all leading to my discussion of procedural flexibility, its relevance in algebra, and I'm trying to make room in algebra for this construct.

A function-based view of algebra is saying that algebra is primarily about relationships between quantities that vary, how to represent these relationships, how to use these representations to analyze, generalize, and make predictions. This is what many people call a function-based view of algebra. So what does that look like? Well, here's a sample problem that I think many people would view within this category or that could be explored within this category, this conception of algebra.

Sarah decides to spend her savings to buy music for her iPod. She has $100 now. She spends $10 each week. How much money does she have in her account? So this is a situation that has quantities that vary. We're interested in exploring and analyzing this situation. We make representations to do so. Those representations allow us to generalize, predict, draw conclusions. So it's showing you a tabular representation, it's showing you a graphical representation. It's all about making and using representations to predict, to analyze, to generalize. That's what a function-based view of algebra is.

On the other side, there's what I call an equation-based view of algebra, which says that algebra is primarily about modeling relationships with symbols, and then using symbols to determine features of these relationships. So note that the equation-based view of algebra is not just saying, memorize these rules with symbols. That's not inherent in the equation-based view of algebra. It's really saying that symbols are important. We use symbols to explore relationships. Let's figure out how to do so.

So for many people, this is a very familiar view of algebra. Here's some problems that we might see in a text that, to a large extent, fall within this equation-based view of algebra. They're just sort of standard problems that are in our curriculum.

Algebra is both of these, but it's difficult. We found it very difficult, where here I mean we as teachers, we as researchers, we as people who write curriculum, we've had difficulty connecting the function- and equation-based views in our instruction. And compounding that difficulty is that these two ways of working and thinking about algebra are separated temporally, meaning that there are times in our curriculum where we're doing stuff that falls squarely in the equation-based point of view, and there's times which fall squarely in the function-based point of view. And they seem very different. To students, they seem like apples and oranges.

And just as an example of that, well, even when they're presented simultaneously, I would argue they're not well-integrated. It's not really even clear what that means. Here's just two pages from the table of contents of a standard Algebra 1 text. And you'll see that in chapter nine, we're doing functions and graphs. We're doing what is a function? We're doing relations. We're doing some graphing. So this seems more function-based in its approach.

You finish chapter nine, you take the test, you turn the page and you're in chapter ten, which is about graphing linear equations, which is very much from the equation-oriented view. So for students, it feels like they've entered another universe. They've switched gears. They see no connection between those two. We try to make that connection, but it's a very difficult one to make, and the texts are not designed to support that connection.

So my work, given this challenge of trying to articulate what algebra is, trying to think about these two different visions of algebra, I've been lead to think about moving beyond questions about what is algebra and what do we teach as algebra, to really think about what does it mean to understand algebra? What does the outcome that I have in mind for this thing called algebra that I want students to really have?

So let me speak about understanding in algebra. And again, I'm leading to the point where there's a place in this question about understanding algebra where procedural flexibility is a good answer.

So I've talked about representations. Typically, we talk about three kinds of representations in mathematics courses, symbols, graphs, tables. Representations are useful for analyzing relationships between quantities that vary. And I would argue that when we talk about understanding in algebra, what we're talking about is two things, the ability to move fluently between representations and the ability to operate fluently within each representation. So it's about between representations and within representations. That's really what we're after in algebra. And let me speak to both of those individually.

So first, I'll talk about, well, let's see. Here's my representations picture. So I've got a table, I've got a graph, I have symbols, and all three of these are, essentially, representing the same quantities that vary. Let me use these three representations to talk about what I mean by between representations first.

So between representations, again, one component of what I think it means to understand in algebra, involves analyzing situations using graphs, tables, and symbols, and making connections between those representations. For example, let's take my three here. Let's say that I'm talking about the concept of slope, a really fundamental, pivotal concept in algebra. If I'm interested in students developing an understanding of the concept of slope, what I'm really after is that they can make connections between different representations which show you pieces of what slope is.

So on the tabular representation, I want students to look at consecutive rows on the table and be able to tell me a story about the slope from the table. And then, I want them to make a connection between whatever they see here and what's going on on the graph. We talk about slope differently in the graph. We talk about rise over run. There's no rise and run on the table, but I need to be able to make the connections. What does rise look like on the table?

And furthermore, there's no rise and run in the symbolic view, but that five tells us something about the slope, and I want them to make that connection. So it's about picturing, finding what slope is in each representation, because each representation gives you kind of a partial picture of slope, and making the connection. And that connection is what we mean by really understanding what slope is. That's between representational fluency.

So I can say the same thing about Y intercept here. The concept of Y intercept is important when we think about functions, and it's sort of partially incompletely presented in each one representation. If I look at the table, I can see, oh, it's the place where X is zero. If I look at the graph, I can say, oh, it's the place where it crosses the Y axis. And the symbolic, I can say, and Y = MX+B . . . that thing after the MX. So those are all little pieces. And by making those connections between the three, that's what I'm looking for about between representational fluency.

So I would say that this is the predominant way that we think about the outcome in algebra that we're after, understanding in algebra. It comes mostly from a function-based view, and this is, I would say, among many people, a commonly shared vision of what we hope students can do.

I'd like to talk now about the within representational understanding part of algebra, because this is the piece that I feel like we don't talk enough about. And this is, for me, where procedural flexibility comes in. So what do I mean by that?

I want students to know concepts and skills for working within a single representation, but I also want them to know multiple strategies in that representation and to be able to select the most appropriate strategies in that representation for a given problem. It's about within a single representation, knowing your way around. And this is what I call flexibility within that representation.

Now many of the examples I'll be walking through with you are symbolic. I wanted to start with one that's not symbolic as a way to make this case, and it uses my tables. So let's say that the problem given to students was to make a table for the equation I've given you here, Y = 1/3X + 4. So imagine what your students would do if had given this problem.

Many of our students would make a table, I don't know if you call these T tables or X-Y tables, and they would just write, okay, 0, 1, 2, 3. And then, they would just plug in and try to get the answers, what Y is for 0, 1, 2, 3 X, because that's what they're used to doing with tables, just 0, 1, 2, 3. Or you might have taught them negative, -2, -1, 0, 1, 2. And they do that for everything.

Well, it turns out for this problem, that's not a particularly smart thing to do. It would be a lot easier if I'm just trying to generate a set of points, it would be a lot easier, since I have a 1/3 here, to pick multiples of 3. It just makes the arithmetic a lot easier. I'm not trying to tell them to never do fractions, you don't like fractions, let's steer clear of fractions. But it's sort of about being strategic here, about being smart. If you're trying to make a table for this particular function, then you should probably pick 3s.

If I change the slope value to something else, I would chose a different thing to make this an easier way. So I want students to, for this particular representation, know multiple strategies, multiple approaches, and to think, before they jump in, and think, well, what would be a strategic or smart or good or efficient way to approach this? And, for me, that's a part of understanding in algebra.

So let me move to some symbolic examples, because I think, especially for those of you who are moving into or working with students in grade 7, grade 8, grade 9, increasingly, we are focused on getting students to think carefully about using symbols. So let's do some symbolic problems. And this is where I'm going to ask you to think about some math problems for me.

So I have some equations that students might see, and I want you to think about how you would solve them. If it's helpful to jot things down, that's great. So here's the first equation.

So imagine that your students are familiar with symbolic approaches to solving equations like this. How would they approach it or how would you approach this equation? I'll give you a moment just to think about this. I won't be taking up your work afterwards and giving you a grade, so it's okay.

Now in a moment, I'm going to ask you to talk to me about how you approached this. But before I do that, I want you to think about a different way to solve the equation that's different than what you just did. Can you come up with a different symbolic way to solve this same equation? And perhaps you can think of a better way, but I'd like you to think about, at the minimum, a different way.

So before we talk about this, I'm going to do something that helps to make my case, which is that I'm going to make a tiny, tiny little change to this equation. And I'd like you to think about what that change does to what you were just thinking about, about the two ways you came up with and thinking about which way might be better. And from a student's perspective, this is a tiny, inconsequential change.

So let's here, the change is, I'm going to change the 15 to a 14, all right? So I'd like you to think about the same thing. How would you or if you want to think about from your student's perspective, how would you solve this equation, and can you think of another way? I'll give you a moment to think about that. So before I show you my next slide, which is talking about different strategies that we might think about for these problems, I'm interested in your reactions, the strategies you used. So for this first problem, 3X + 1 = 15, how would your students approach that, typically? Yeah?

WOMAN: (inaudible) first, and then . . . through all the dot sets too . . . line . . .

DR. STAR: Good, good. So they distribute first, and how many of you would agree that that's probably the way your students would approach this? So why do you think that that's the way they all would approach it?

WOMAN: (inaudible)

WOMAN: (inaudible)

DR. STAR: So that's how it's taught. Do you feel that you might have a picture in your mind of pages from the text? Certainly, there are texts that teach this as the way, the only way, the best way, let's not talk about any other way to solve this problem, right? But it isn't the only way to solve this problem. So when you were coming up with an alternative way for this one, what was your alternative? Yeah?

WOMAN: I teach my kids to undo it . . . there, they're multiplying by three, so you've got to over here. And you're undoing it, you've got to divide by three. And then, you subtract the one.

DR. STAR: Good, good. So that's an alternative, right? Whether it's sort of couched as an undoing or whether it's just let's divide by 3 first, sort of as a step then, that's another way to do this. And when I change the 15 to the 14, which, again, is a very minor change that students barely even notice, potentially, when I make that change like that, how did that affect the relationship between these two strategies from the mind of the student? Yes, in the back?

MAN: (inaudible) the first . . . maybe that's why the first . . . because it's nice if get things to go in . . . 15 worked out real nice . . . regardless of . . . that's a nice coincidence that . . .

DR. STAR: Excellent, excellent, yeah. So we'll definitely want to come back to that in a minute. Yes?

WOMAN: Well, some of the students . . . if it just . . . and then, if I turn around and do the undoing, then they go back to the first method again. And then, somehow they understand that better. I don't know if it has to do with the students that I have or whatever, but I have found that to be true, especially with special ed students . . . students. I work . . . so oftentimes, they also . . . because they're like, oh, I did this, you know, myself. And then . . . you know. So then that way . . .

DR. STAR: I like that. So that really says some of the points we're going to talk about in a moment about knowing multiple ways. But let me just put together what you just said. It relates to what I have here.

So as you said, this is a very common way, many texts teach this as the way, the only way. It's what students sort of jump into and do. But as you pointed out, there is another way. And arguably, for this problem, that second way is better. We could talk about what better means, but, in this case, you could say, well, it has one fewer step. In the discipline of mathematics, we might start using words like elegant, which is a word that mathematicians use a lot. So we might say, it's more elegant. And it's hard to define what elegant is, but maybe elegant is that the solution method is optimally matched to the problem features. That might be one way to think about it. But it's better.

But as you pointed out, the second problem, we did use the first strategy exactly as it was. We could use the second strategy exactly as it was. But I think we can make the case that the first strategy actually is better for the second problem because you're going to have to deal with fractions in this problem, no matter what. But you kind of minimize their use of them, in a way, with the first strategy, and that seems to make it better.

So this is getting in the universe of what I think about with procedural flexibility. And it's worth stepping back for a minute and asking the sort of so what question, because this is where this often comes up. As the gentleman in the back pointed out, the first strategy works for all problems.

So someone might say, well, isn't it better to just only learn that one strategy? It works for all problems. It's sort of a waste of time to learn these other strategies that sometimes work or sometimes helpful, for sometimes not. Because then, you've got to figure that out. You've got to look at the problem and you've got to say, well, of all the different ways I know, which one should I use? Why don't I just teach them the one way that always works, and they'll just know that way and always apply it?

And that's actually what I'm pushing against, that I don't agree with that sentiment. I argue that we want students to know multiple strategies in this representation and the ability to select the most appropriate strategies for a given problem. That's what we just illustrated here.

If they can choose the most appropriate strategy, what that means is that it actually reduces error. This isn't just, oh, I'm going to play this game, and I want to solve it in the fastest way, and that's the game I'm playing. But if they choose wisely and strategically about a strategy, it's going to make it more likely that they're going to get the problem right. And it's also going to allow them to do it quicker, which does play a role. We are interested in them getting accurate, quick solutions for problems, especially in today's world of standardized testing.

Furthermore, and this is what I do research on, one might say, well, okay, I want them to know multiple strategies, but how is that linked to this understanding thing that we're really after? And it is linked. We found this as linked. If you're interested in conceptual understanding, then students who know more than one strategy and can pick and choose, they have greater conceptual understanding than those who don't. They get it.

And furthermore, well, with respect to the studies that we've conducted on this, the conceptual knowledge that students have in this particular problem domain has to do with equivalence and variables. When students know multiple strategies and can use them, then they actually understand a lot more about the concepts of equivalence and variables than those who just only know that one strategy and always apply it. Yes, question?

WOMAN: (inaudible) level, but . . . elementary level math . . . that's exactly what we're doing. We're showing the kids different strategies . . . and then, it's whatever they feel comfortable with to use that strategy. And it's really . . . our math scores have really gone up. They're really doing very well . . .

DR. STAR: That's great. Well, I would say that, in some ways, this is true, that there's a, the spirit of this does come from what we've been doing in elementary school, that I would say more typically in elementary school, we've been asking students to come up with multiple strategies for a long time and with great success. And at the secondary level, we've been resistant to doing so for, I think, interesting reasons that I think a lot about.

Let me put out what I think is an important difference between what you often would see how this is done in elementary versus in secondary that I think we'll come back to in a little bit. But in many cases that I've seen in elementary school when teachers are getting students to think about more than one strategy, then the endpoint might be that the student is going to look at the multiple strategies and see which one makes the most sense to him or her. And then, that's the one they're going to use.

So I'm going to sort of show them lots of different ways or they're going to show each other lots of different ways. And at the end, I want them to kind of latch onto one, and that's the one they're going to use.

And I think I'm putting out a different kind of vision, which I think is a little more relevant to middle, secondary school, which is that I'm going to put multiple ways out there for everybody, and I actually want them to know them all. I don't want them just to pick the one that makes the most sense to them, and always use it, and never deviate, because I don't think that's much different from what we do now, which is, we tell them the way that they should know. I want them to actually know lots of different ways, and to pick and choose which way is most important.

So, for me, it's about showing them lots of ways, and they need to take away lots of ways. It's not about showing them lots of ways and them taking away one way or the way they most prefer. And I'm not saying that's what you do in your elementary curriculum because there's a lot of variation in that, but I think, stereotypically, that's a difference that I try to key on in elementary versus secondary.

WOMAN: As they hit 12, there will be math problems, and it's tough not to go do it one way. And they say, well, it will be easier if I, you know, add up . . . and it's nice to see that they do have these options . . . math problem, it's whatever is starting to make . . .

DR. STAR: That's right.

WOMAN: (inaudible)

DR. STAR: That's right. So it might not always be that the, their criteria for choosing might not always be exactly the same as my criteria for choosing, but at least they're having that conversation. They're saying, I know several ways I could do this. I'm going to think for a moment, before I start, about what's the best way to do it for this problem. And that's the thinking that I think we're after. For me, that's an indication that they're flexible. They're trying to show flexibility. Yes?

WOMAN: I just wanted to make a comment. I'm the math person . . . and I just wanted to say like I would . . . but I think it's important to understand that there's multiple strategies . . . when you go to other disciplines. Because, you know, of course, that's a symbol-kind of thing. And, of course, letters are symbols that represent words. So knowing multiple strategies can help you in other disciplines.

Okay, like how do . . . you don't always have to use the way you're always taught. So I think it's important how, like, okay, using algebra, people say, how do you use it in everyday life? Well, it doesn't just apply to mathematics. We use those types of strategies in other disciplines.

DR. STAR: That's a nice point. So in some ways, this is sort of a mindset that we're trying to get students to have about what it means to learn math that is a similar mindset to what's applicable in other disciplines, that I, thinking about the strategies for solving problems, and it's not always, my teacher is not always primarily, solely, exclusively interested in the right answer.

The teacher is also interested in how I did it, and I may have to explain and justify how I did it, because that is an indication of my thinking. And they're evaluating my thinking, not just if I got the right answer. And that's sort of where we're heading here. It's about a way of thinking differently about math, that it's about how I do it, and being accountable for my decisions and my choices in strategies.

So let me do another example here. I think when I'm speaking to some audiences, particularly if I am in a room of high school teachers or mathematicians, this particular example falls flat for some of them because they think, well, that difference between the first and the second strategy, well, that is so minor. That is so tiny. It doesn't really matter. Even if they get their head in the minds of their algebra students, they would say, well, that's such a minor savings, if you will, to do it that other way that is it even worth knowing that? So let me come up with a different problem that I think is a better indication of what I'm after here.

So here's another problem I'd like you to think about. This is your garden variety proportion problem, solve for X. Now I'm going to play a similar game to what I did with you last time, which is, I'm going to make a seemingly minor, inconsequential change to this problem, and I'd like you to think, again, how you would solve it.

And so, the change I'm going to make is, I'm going to change the 7 that's under the 16 to an 8. So otherwise, it's exactly the same problem. I just changed the 7 to an 8. I'd like you to think about how you would solve it. So I'm interested in how you think your students would approach either of these two problems or how your text teaches students to approach problems like this. Yes?

WOMAN: My students . . . would take the first problem and say, it is 32 . . . 7 x 14 . . . 32. The second one, they wouldn't, some of them might come up with 56 and then say, oh, 4 times that is . . . times 7. But I think cross-multiplications . . . students with . . .

DR. STAR: Good, good. So you can let me know if you agree or disagree, but I think that cross-multiplication, as she pointed out, is a very common way that students are taught to solve problems like this. In many texts, it might be the way, the only way, the way that is taught. Some people have concerns about cross-multiplication because they feel like it doesn't encompass understanding what you're doing. You're just sort of going through some steps, and it's sort of separated from those.

And yet, this is how we teach students, to teach problems like this. But as you heard in the response here, it's not the only way to solve problems like this, and there might be other ways to think about problems, given their particular arrangement here.

So the reason I think this is important is, if you think about the problem, as you mentioned, cross-multiplication is a common strategy here. So if students are doing cross-multiplication on this problem, they would just say, 7X is, and then, they would sort of say, 14 times 16, uh, let me get at my calculator. So it's a total bump in the road, right there, with cross multiplication, right? They're sort of stuck, and they've got to figure out, you know, I need some help figuring out what 14 times 16 is.

There's another strategy here, which the lingo for people like me is, this is called the equivalent fraction strategy. And so, what we would say is, okay, as you said, 7 times 2 is 14, so 16 x 2 is 32. That is so much easier. It's faster, they're less likely to make a mistake.

So with the second example, I just changed the 7 to the 8. And by making that small, small change, then cross-multiplication, sure, it works exactly the same, right? I say, 8X is, and if I hadn't done 14 and 16 before, I would have the same problem. The 14 and 16, what? What's that? Let me get at my calculator. Let me calculate that. It's a place where students get stuck, it's a place where error comes in.

Because of the particular arrangement of numbers in this problem, there's another strategy which we might call the unit rate strategy. And that's where you say, well, 8 times 2 is 16, so 14 times 2 is 28. That is so much easier. It's faster, it's less likely for them to make mistakes.

So the real hard part here is that students need to know all three of those strategies that I just showed you, and then, they need to look at the problem features to figure out what strategy is the one they should invest in for that particular problem. So for the first problem, maybe we could hope that they look at it and they say, well, I know cross-multiplication. But, you know, equivalent fractions is going to be better here.

For the second problem, they would say, well, equivalent fractions doesn't do much for me here or that's going to be hard. I'm going to use unit rate. That seems to be the easiest. So that's the thinking that we want here. That's hard, but that's what flexibility is about. So what I've been talking about here is this, oh, yes, question?

WOMAN: Not the strategy that our book shows us . . . because the 7 was there, and . . .

DR. STAR: That's nice. That's a great one too.

MAN: (inaudible) 16 . . .

DR. STAR: To . . .

MAN: (inaudible) over one. And then . . .

WOMAN: Yes.

DR. STAR: That's nice. So what I find interesting when I talk about this with folks in the math world is that this construct of flexibility that I talk about, pretty much, we all have it. We got it. You know, you can tell me all the cool, different ways, and it's not because, I find that math teachers are really good at telling me wacky strategies that students use because we've seen them. You're not going to believe what my kids did on this problem, in sort of a bad way. So we have a knowledge of wacky strategies, but we also have this deep knowledge of efficient, clever strategies for all these problems like this.

And as I'll talk about in a minute, we may have this desire to communicate that to students, but I'm not sure we're doing it in the best possible way. So that's what the remainder of the talk is really about is, instructionally, if this is a goal, how can we do it, so that students will have this knowledge about multiple strategies? I saw another hand. Yes?

WOMAN: (inaudible) for our kids is that they've been shown . . . so for that second one, my kids know that, you know, fractions can . . . division . . . so they would say . . . I need to reduce it now to two. So what's 14 . . . equivalent? But they're doing . . . was saying, her doing . . . but hopefully, when they get to the high schools in our district are already . . .

DR. STAR: That's nice. So it's another point worth taking, for those of you who work in middle and high schools, is that these sorts of things, the nuggets of these sentiments, are happening in many elementary schools, and I'm not sure we're making use of them as much as we can in the high school. So just, yeah, Jess, go ahead. Sorry.

JESS: So when you're teaching all these strategies to my kids, does it work to have names for each strategy, so that the students can identify those? And, if so, like is there a source where it's common names for . . . out there, and like people have access to it?

DR. STAR: That's a great question, and it does relate to some of the things I'll talk about in a minute, but let me give you a brief answer now, which is that, I would say the answer is, yes, it's important to have a name. But I don't think it's critical, for the most part, to have a standardized mathematically agreed upon name. And this is another place where we look to what's happening in elementary school.

In many elementary school classrooms, we might do a problem. And Susie, over there in the corner, it's her turn to share her strategy. And she goes to the board and she does it this way, that I, as a teacher, look at and I like. And I say, let's call that Susie's strategy. And we may refer to that as Susie's strategy for the next month. It may be called Susie's strategy.

Now there may be a point later on where we either give it a formal name or we just stop calling it Susie's strategy, but all we need is a way to talk about it. This is about getting kids to talk about the strategies, so that they can compare and contrast them. So they need a way to refer to it that we all are on the same page on. And if it's Susie's strategy or if it's equivalent fraction strategy, initially, it really doesn't matter. It's just, it needs a label, so we can refer to it as an object. We can talk about it. We could say, remember Susie's strategy? Would it be useful here? And this is going to come up in the way I'll talk about the instructional interventions in a minute in a big way.

So just to step back for a moment, the universe I've been trying to create here is, we were talking about understanding in algebra. I talked about between representations, which means, you're trying to connect graphs, and tables, and symbols, and that's really important. But I've been talking about, within representational fluency, as where procedural flexibility lies. Within representational fluencies is where we want students to know multiple examples, multiple strategies to approach a class of problems, and then, to choose the most appropriate strategy for a given problem, where appropriate can mean lots of different things here.

This moves students beyond, in my opinion, rote application of a single strategy or single procedure to more flexible use of multiple strategies, which I think is a very important outcome, especially in algebra. And this is what I call procedural flexibility.

So let's say that, okay, Jon, I'm with you. This is a good thing to do. How do you do this or how can we develop this in our students? Well, that's a hard question. And yet, if I were, you know, king of the world here and trying to think of a good answer to that question, what I would want to is think about ways that teachers could do this in their class that are easy for them to do and have a big impact on their students' learning.

A lot of the things that we, on this side, ask teachers to do are really hard to do. And they have to spend months learning how to do them, and it's really tough. And then, the next year, we ask them to do something completely different. And it's not clear that they really have a big impact anyway, even if the teacher invests in them. It's a challenge.

So in my work, what I try to do is come up with things that I think are easy for teachers to do, but I think have a big impact on their students, and that's what I'm going to talk to you about. What kinds of small changes in teachers' practice appear to have a significant impact on students' flexibility, particularly in algebra? And the key thing that underlies all the interventions that I'm going to talk about, all the little things I'll talk about, it's comparison. And they all draw on the benefits of comparison.

So comparison is how we learn. It's a fundamental learning mechanism. We learn by comparing things, as I'll illustrate in a few minutes. If you look to the research world, there's a lot of research evidence that supports the power of comparison from psychology, from cognitive science. When we compare things, as you'll see in a moment, it helps focus our attention on critical features of what we're trying to learn. So I'm going to take you through an example now to try to illustrate this, and I'm going to choose an example that's not in mathematics to try to make this point.

So the holiday season is over, but likely you were in a situation, not too long ago, where you were buying something online. And so, let's imagine that I am trying to buy a camera online at Best Buy. Are they still in business in this, yeah.

WOMAN: (inaudible)

DR. STAR: It's not doing so well. So let's think about how I decide what camera to buy and how comparison helps this process. And just for the sake of argument, I'm going to simplify the universe slightly. If you ever try to do this, you'll see there's hundreds of cameras to choose from, so it's a really daunting task. I'm going to simplify the universe slightly by saying that I'm going to start with a price, I'm going to pick two cameras at that price that both seem to look okay, and I'm going to think with you about how I might decide which of those two cameras I should buy.

So let's say I'm looking at, and these are cameras that are probably not still sold a year or two ago. So this is the SD30 that's $300, and the SD1000 which is also $300. So how do I decide which camera to buy?

So one way that I would do this is, I would just do my homework. I would do some research on the website, right? So I would say, okay, let's look at the SD30. So let's do that. So here is the website for the SD30, and I'm going to just do my homework here. I'm going to read all about it to figure this out. Okay. And I should say, by starting off, that I really know nothing about cameras.

So, okay, it's five megapixels. I'm not sure what that means, but I'm going to try to remember that. It's five megapixels. Okay, let's go down here. It's got a lot of features. It's got automatic noise reduction. Hmm, that sounds good. All right, let's keep going. Okay, this seems helpful. Look at this. There's a list of features. I think this is going to help me.

So I know that you can't read this super clear from the back, but it's a list of features. Let's see. I see that, okay, there's my five megapixels. There's a difference between effective and total. I'm not sure what that is. It has a digital zoom of 4X. Okay, I think that's good. It has a burst mode. I don't know what that is, but, okay, it has a burst mode. And it has audio, it's got two video outputs, okay. So I just did a lot of, sort of looking at this particular camera. Great.

So then, I would say, okay, let me look at the other camera and see if I can see if that one might be better. So here's the other camera. It's got 7.1 megapixels. That's more, because I remember the other one was 5. So that's more, okay. So that seems to be a good thing. All right. It's a lot of things here. This records QVGA video clips. Now I don't know if that's good or not, but I don't remember if the other one did or not.

Okay, so here's my list of product details. This one is sleek and slim, that's a good thing. And this also has a burst mode, I see, which I'm sort of drawn to, just because of the name there, burst mode. I don't know what that is, but that sounds kind of interesting. So it's a long list of features that I can look at.

And I would say, after looking at this list of features, I have no clue which camera I should buy. Now, why? What I just did, I call that sequential viewing. So what I've tried to illustrate there is it is really hard to remember information about the SD30 when looking at the SD1000 and vice versa. It's just hard to remember. There's too much.

And even more importantly, it's hard to know which features the cameras are similar on and which they're different on. That's really what I should be paying attention to, if I'm going to make a decision, how are they similar, how are they different? As a result, if I'm making a decision like this, I am just as likely to be influenced b superficial features, like shape and color or burst mode because it sounds cool, than by substantive features that really matter for this choice.

The answer to how to do this better is to compare. And even without reading my research on this, Best Buy knew to put this in their website to allow you to compare. They have a way, and lots of websites do, where you can put things side by side and compare them. So here are the two cameras, side by side. So I can just look down, and I can see the two megapixels difference, right there. I can see that digital zoom, hmm, they're the same. It both says 4X digital zoom. So I can forget about that. It doesn't matter.

Okay, burst mode. Hey, they're both the same in burst mode. I can forget about that. It doesn't matter. So I can look down and say, okay, they look to have, let's find a place where they're different. Okay, oh, camera dock included, camera dock not included. I can see a place where they're different, and then I can say, is that difference important to me or not? So by comparing, it makes it easier for me to see the features where these cameras are the same and different. And I would argue that this exact same benefit of comparison applies to the way students learn math.

If we ask students to compare, it helps them see the similarities and differences between problems, representations, and strategies. That allows them to see how and why a strategy works for a given problem, as I'll show you in a minute, and it helps them see how and why a representation is particularly useful for answering a certain kind of questions. Furthermore, they see why some strategies are better than others on some problems, which is what we're after when we think about procedural flexibility.

So I want to illustrate these by talking about three little things that we found in our research that teachers can do that do really help, that draw on comparison, and that are especially useful at promoting procedural flexibility. And I'm going to just list them, right now, and then, I'll go through each one in a little more depth.

So the three little things are alter instructional presentation of problems and strategies, so that comparison can occur, engage students in conversations focused on comparison, and provide opportunities for students to generate multiple approaches, so that comparison can occur. So let me talk about these three individually.

So the first we found is among the most powerful, and it's really easy to do. It's that instructional presentation, when we present things to students, it should provide opportunities for them to see multiple problems and strategies at the same time, rather than sequentially.

So what that looks like is that we could present them multiple examples at the same time on the same page, for example, or we could present them examples that are on separate pages. And we found that the one on the left is much better. Now the hand motions for this are this versus this. So this is what we're after here. I'll show you what this looks like.

Now this can take several formats. We can display the same problem solved two different ways, side by side. We call this, in our research, compare strategies. We can display two different problems solved the same way, side by side. We call this compare problem types. Both of these ways of presenting problems are better for student learning than sequential presentation.

So what does this look like? Stereotypically, if I walk into a math classroom, present company excluded, what I would probably see is the teacher would show students an example. So, hi, class. Let me show you about this problem. I want you to think about how to solve it. So here, I'm doing it on the board. Here's the problem. This is the one you solved, by the way.

So here's the problem, I solved it. I might get students to ask me questions. I might have them come to the board. I'd do this. So I say, any questions? Everyone all set with this problem? All right. So what do I do next? I erase the board, right? So I say, here's another problem. Let's think about this one. That's the second problem you solved.

Note that it's the same problem, all right? We knew that because we did this a few minutes ago. But it's really hard to compare this to, the students likely have no idea that this is the same problem. It's hard to compare the two strategies here when I present them this way, and that's what I started to illustrate in the camera example.

Alternatively, if I present them together, like that, like you saw, they're both on the board at the same time, we can compare them much easier. I can say, hey, they're the same problem, but you solved it different ways, and one way looks shorter than the other way. I wonder why?

In particular, when we've done this in our research, we've found that students, when you put them side by side, start to see that that step there, where I divide to both sides, is mathematically the same as that step. Mathematically, they are the same thing. But when you do it on the problem on the right, it really helps you solve that problem in a different way and a faster way, a better way. It's the same step, you just are using it in a little different way in a different problem. And that's what we want them to see.

And furthermore, in our research, when students were given these things side by side versus sequential, they were more likely to adopt that more innovative strategy when we gave it to them, just solve this problem later on. They were more likely to use the strategy.

So this is the first instructional recommendation that our research has found is powerful is, just putting things side by side and giving students the opportunity to compare. And in a little bit, I'll show you what this looks like in the materials we've developed.

With all of these recommendations that we found are effective, I'm going to offer some cautions about their use, however, because all of these things, just like anything else we're doing, it has the potential to go awry. So I want to put some cautionary note on it. The caution with this particular one is that the things that you're comparing side by side, they need to differ on some important dimensions, but not be completely different. You have to manage the degree of difference. And how the examples differ needs to be important and relevant to the problem-solving process.

So just to illustrate these, if I were in my Best Buy example, and I put a camera side by side with a stereo, that really wouldn't help too much, because they're different on every dimension. So just putting them side by side doesn't help. They're too different. I have to manage the degree of difference, so that they're similar enough, but somewhat different, and the thing that they're different on is what I want students to notice, because that's its power.

Furthermore, if I were doing my equation solving example that you saw, and I put two problems side by side, and one of them I wrote in blue marker and one I wrote in red marker, but otherwise they were the same, I can guarantee you that students would be more likely to notice that difference, if they're like this, than they're like this, but so what? So the thing that we have to, in this recommendation, it's about carefully deciding what you want the difference that's noticed to be and select problems that the comparison highlights that difference for.

So the second and related instructional intervention that we found to be effective has to do with what we call comparison conversations. We haven't found evidence, although we haven't looked for this, specifically, that merely putting things side by side versus sequential, and having nobody talking about anything, helps all by itself. But rather, we found that, if you combine this with providing students opportunities to compare and evaluate different strategies, that's where the benefit comes in. So it's putting things side by side and giving students an opportunity to process that, and talk about it, and have a comparison conversation.

I'll talk about two different types of conversations that we found to be useful here. One has to do with similarities and differences, and one has to do with evaluation of strategies.

Similarities and differences, when I put things side by side, I'm going to be asking students questions like this. How are these strategies similar? How are they different? How are these strategies related to other strategies you've used before? That's about similarities and differences.

I could ask the same questions about the problems. I could say, how are these problems similar? How are these problems different? How are these problems related to other problems you've seen before? But it's about getting them to think about what's similar and what's different about the things that I'm comparing. Because I've chosen the example so that what's similar and what's different is important, and I want them to see if they can notice it.

For evaluation, I'm going to be asking questions like, which strategy is better for this problem and why? Why is this strategy the most effective or the most efficient or the most elegant or the best for use on this problem? And you can start to see why this is where it's useful to have a name for the strategy, so you're not saying this strategy, that strategy. It's why is the equivalent fraction strategy better for the problem on the left?

And this is a question that I really like that's a hard question, but if they can answer this question, they got it. On what kinds of problems is this strategy most effective or most elegant or best? That's a great question.

If you think to our proportion example, if the student says, well, if I can see that the ratio on the right, if I can see that there's one number evenly divides the other number, then I kind of think that strategy that does unit rate is going to be the best. But if I can see that the denominators are actually multiples of each other, then I think that the equivalent fractions is going to be the best. If they can tell you that, they got it. So that's a great question for trying to see if they see which problems, the link between problems and strategies that's most effective. Yes?

MAN: (inaudible) of when . . . first represent . . . seems to be . . . the first step is to decide . . . quantities . . . have differently worded problems and then show how they compare them, you know . . . and then . . . to solve those problems . . . but that seems to be, once you get . . . getting to that part, representing the . . .

DR. STAR: That's right.

MAN: (inaudible) that flexibility . . . is also important . . .

DR. STAR: That's a great point. So I agree that that step of, some people call it translating or representing the word problem as given into something that you can then work with, is a very difficult step for students, and that there are often many ways to do so. And I think comparison might be useful in illustrating different approaches.

I would say, we haven't looked at that particular use of comparison for a couple of reasons. One is that we've been careful to try to manipulate the degree of difference, as I talked about earlier, so that the thing we're comparing is not too different. And sometimes when students are approaching the translation of a word problem in very, very different ways, then it's harder to see how comparing them would, how you'd see similarities and differences because they're so different. That's the first thing.

And I guess the second challenge that is worth thinking about for using comparison in that setting is that in our problems, we try to make it so that there's, essentially, there's a message that one takes away from the things you're comparing. So in the examples that I've given you, the message is about this strategy is easier for this kind of problem. So there's often kind of an easier/best-kind of message. It's not the only message one could take away.

And I think for some word problem translations, you can't really make that claim. It's hard to say, well, this way is probably easier or better. Either is okay. And so, I think it's worth thinking about how you can make use of comparison for that, but I would leave those caveats on the table as challenges.

So that relates to kind of the cautionary notes about this recommendation. So we're talking about comparison conversations, getting students to compare and contrast strategies and problems, and talk about similarities and differences. So the cautions with that are, I caution you from doing this in a way that leads to what I call serial sharing.

So serial sharing, stereotypically, is when I might say, okay, you know, we're working on this problem. Johnny, how did you solve it? How did you approach it? And Johnny tells me his way. Great, Johnny. Excellent. Susie, how did you do it? Susie talks for a while. Great, Susie. Nice. Jimmy, how did you do it? He talks. Great.

So I'm not comparing. I'm just getting people to sort of tell me what they did in a serial way. It's sort of like a, sort of a show and tell-type thing, where everyone is just telling me how they did it, and that's not comparing. Essentially, you have created an environment where you could compare, because there's multiple strategies out there in the ether, but you haven't taken advantage of that opportunity. And, for me, that's serial sharing.

The second caution on this is that comparing is not easy to do, especially when we're looking at complicated things or new problems. Teachers need to provide visual and gestural cues to aid comparison, that they might use especially in these conversations.

I don't know if you're familiar with the TIMM studies. It was an international comparison study where it's the one that, often in the newspaper, you'll say, it's the one that says, U.S. students did terrible, blah, blah, blah. It's one of those.

So when various people looked at instruction in other countries, especially countries that were high performing, and they looked at this particular issue, they found that, in the high performing countries, when comparison was occurring, and in the high performing countries, they were doing a lot to help their students compare.

They were doing things like, I'll do it with my little pointer here, but you can imagine with my hands. They would say, see that step, right there? That's the same as that step, right there. They would sort of do this a lot, point, gesture, use visual cues, use gestural cues. They were helping students attend to the thing that they should be comparing. And we need to do that.

So if a student, you know, I've got my two problems on the board, and I'm having this conversation, and this. And how are they similar and different? And the student says, well, they next to last step is the same. In other countries, we would see the teacher going up and point. You're saying that this step is the same as this step? And that really helps everyone see what the student who saw that is comparing.

So we have the side by side presentations, we have the opportunities for comparison conversations, and the third is about multiple approaches, and it's about that instruction should provide opportunities for students to generate multiple ways to solve the same problem. And this is something that I think is related to the other two, but it's something that I don't think we do a lot of, typically. This can take several different formats.

Students might be given one solution to a problem and asked to generate another different one. That's sort of what I did with you. I gave you a problem, I asked you to solve it, and I said, hey, can you come up with a different way to do it? I could give you a problem and ask you to do it in two different ways, right off the top. Or I could have a whole class discussion where multiple different solutions are generated. So I've got a couple that came out, and I might ask a question like, hey, can anybody think of another, different way to approach this problem that we haven't seen yet?

This act of generating, getting students to generate multiple strategies, it's linked to comparison, because the act of producing solution methods that are different, even on an individual level, requires you to compare. So when I gave you a problem and you solved it, and then I said, can you do it in a different way? You had to think, well, what is different here? I know this other way. Is that different? Let me think about those two ways together and think, well, is that different? How is it different?

So the act of producing a different solution involves some comparison. And furthermore, if I get students to produce multiple strategies in the presence of those multiple strategies, it links to the other two recommendations. I can put this one side by side, I can have a conversation about the similarities and differences, and the evaluation of those different strategies.

So this recommendation is the one that has cautions. And, in some ways, these cautions are especially important. The number one caution about this is, this can't be busywork. So imagine in my class, the students just spent 20 minutes solving a problem that I just gave them, and it was really hard, and everyone feels happy they got it. And I say, okay, turn around and do it another way now. They're just going to go, uhh.

So it can't be busywork. I have to have a reason for doing it. It's not the mere generation of the multiple methods. That's not the task. The task is rather to, once we have the multiple methods, to have a way to talk about them. I'm trying to generate the raw material for a comparison conversation. And the way to do that is to have students generate and to think on their own about what's a different way to do this.

And furthermore, the differences, if I ask students to generate a different way, I have to be prepared for the fact that the differences in their methods may be trivial or not relevant. If I'm telling them to produce a different way, then I have to be willing to accept the fact that what they produce as different is not what I would categorize as different, but they came up with it as different.

And so, this is yet another reason why this task has to be combined with some way to discuss. So, for me, a good example of this is if you think of a linear equation that has variables and constants on both sides of the equations, so $2X + 10 = 5X + 7$. So that's a standard equation they see early on.

So if I have students produce multiple ways on that, then a student might say, well, on this method, I put the variables on the left side and the constants on the right side. And on this way, I put the variables on the right side and the constants on the left side, and those are different. So that is a great conversation to have.

On the one hand, I could be the thinking, well, that's kind of trivial to the solution. I'm not sure that really leads to a different or a better solution. But I'd like to find out from the student whether they do or they don't. And furthermore, I can actually think of a problem where that actually might matter.

Like if the coefficient of the variable term turns out to be negative, then I might want to move things to the side where the negative doesn't appear. So those of you have taught this, I know you know what I mean. But if I move to this side, I end up with 3X. And if move to this side, I end up with -3X. And it's a little easier if I just do the 3X. I'm not going to lose that negative sign. And that's a reason to have this conversation.

But the point here is that, if I ask students to generate a different solution method, I have to be prepared for the fact that what they may come up with could be seemingly trivial, and that's part of the conversation that we need to have is, does that difference really matter?

So the bottom line for these recommendations is that there are three relatively small things that teachers can do that we found do have a significant impact on students' learning, particularly in the realm of algebra and particularly related to the development of procedural flexibility. They were showing examples side by side, discussing and evaluating multiple strategies, and having students generate multiple ways.

So before I think a little more about how this looks like a little more specifically in practice, just to sum, the point here is that it pays to compare. Comparing is how we learn, and we can do a lot more in our math classrooms to make use of the power of comparison. And, in particular, comparison is linked to gains in procedural flexibility in algebra. I talked to you about these three ways.

The work that this is based on, we conducted several years ago in small studies, where we were in classrooms for a very short amount of time, like a week in a class, where we were doing something really constrained. But we found that things worked, and we set out to do something much bigger and much longer. And that's what I'm going to talk to you a little bit now, because I think it's especially relevant to you thinking about you using these recommendations.

So what we've done is, we've developed a set of what we call supplemental curriculum materials, which we call them worked example pairs, because they are solved problems, placed side by side. And they draw upon these interventions. And we're involved in a big test right now in Massachusetts, where we're seeing whether these have the effect on student learning that we think they do.

And so, basically, what happened is, we got a bunch of teachers together and we decided half of them were going to use our approach and half were not, but use our approach the next year. And we had a week-long professional development last summer, and we helped the teachers see what our approach looked like, and we talked a lot about comparison and how our materials made use of comparison. The teachers are using our materials. They're videotaping themselves. We're tracking their students, we're tracking their instruction, and we're looking to see how this plays out.

So I wanted to show you a little about what our materials look like and features of our materials, because I think they were trying to instantiate the recommendations that I just told you about. So there's a lot on this next slide that I'll try to talk through, but I'm going to show you some examples of our materials.

This is a little small. This is a single piece of paper, a page. It has cartoon characters which have, this is Alex and this is Morgan. So Alex and Morgan are the primary actors in our materials. And Alex and Morgan typically are given a problem to solve. So in this case, it says, Alex and Morgan were asked to graph the equation Y = 1/3X + 4, using a table of values. This is the table with an example I showed you.

So there's a label up here. It says, Alex, choose typical X values way. And this says, Morgan, choose X values more carefully way. Alex does it, here's his math. And then, he has a little dialogue bubble where he says what he's doing. Morgan does it her way, there's her math. And then, she has this dialogue bubble which tells what she's doing. And there's this line down the middle, so we've got the side by side.

And then, at the bottom, there's some questions that students may consider, that teachers may conduct a discussion around. It says, how did Alex graft the equation? How did Morgan graft the equation? So the first question is just seeing, can you figure out what Alex did? Can you figure out what Morgan did from looking at this? And then it says, what are the similarities and differences between Alex and Morgan's way? Why did Morgan choose to use only multiples of 3 for X? And then it says, which way is easier, Alex's or Morgan's? Why?

So our vision is that teachers would use something like this to talk about this particular problem or this idea of making tables for graphing equations. They would put this up on the overheard or on the SMART Board. They would give students time, usually in small groups or in pairs, to think about it first. Then they would engage students in a conversation.

Typically, for our teachers, it might mean having a student come up here and tell me what Alex did or be Alex. And then, another student come up and tell me what Morgan did and be Morgan. And then, we would have the similarities and differences conversation, where I might solicit students to say, what are the similarities and differences? Which one is better, and why? So there's lots of different ways that teachers have a conversation around this. There's considerable instructional flexibility in how one would use this.

So we'd have a discussion. The whole interaction could take 5 minutes, it could take 30 minutes. And then, at the end, we put up this second page, which is sort of our little splash take-away page, where, in this particular case, Alex is, they're trying to tell you about what we should take away from what we just spent 5 or 30 minutes on. And so, Alex says, when creating a table of values to graph an equation, it's helpful to choose X values that will generate whole number values for Y, rather than simply choosing X values, without considering the specific equation to be graphed.

And Morgan says, huh, interesting. There's more than one way to pick points for graphing a line. Before you start picking points, you can try to look at the problem first, and then, try to pick the points in the easiest way. So this is what we created this example to try to illustrate, and Alex and Morgan are just kind of encapsulating it. In the same way that in any lesson I'd be teaching, I would sort of have a point of closure, probably, and say what the point is, and this is the point.

Ideally, in the discussion that we just had, this came out. They've already heard this. And I'm just saying, okay, let's see what Alex and Morgan have to say about what the takeaway was from this. And, look. They said the same thing we just said. Joey, can you read what Alex said here? And, Susie, can you read what Morgan said here? And that's the way this ends, typically.

WOMAN: So what is this called?

DR. STAR: Well, we call each of these a worked example pair. And together, it's just our curriculum. I'm not sure we have a name, the Alex and Morgan stuff.

WOMAN: (inaudible)

DR. STAR: Yes?

MAN: What do you do when a student puts something out that's incorrect . . .

DR. STAR: Yeah, that's a great question. So it's really hard to see, but at the very top here, there's a question. It says, which is better? And that appears at the top of this particular example. And that's because, for this comparison, our instructional goal was that, by comparing, you got some information about which one was better. That's the point here.

But we have, depending on how do you count these days, about four different flavors of this. They're not all, which is better? We have different goals. So there's a flavor that's called, who is correct? And so, depending on the problem, Alex might be right and Morgan might be wrong, or vice versa. And it's not clear, when you first put it up there, who's right and who's wrong. So that's one of the questions.

It might say, how did Alex solve the problem? How did Morgan solve the problem? Who's correct? How do you know? So we definitely put right and wrong answers side by side, using the exact same thing. It's the same kind of conversation. Because that's something that, when you put them side by side, students can notice.

And furthermore, if they notice that Alex and Morgan got a different answer, and then, they have a way to figure out who is right, then they can look back up at their work and say where they were different, and see whether that is what lead to the error. They can say, oh, they were the same on all these steps, but this step is the step that they differed on. I bet that's why Morgan made the mistake is because she did this step wrong. We know that students have a really hard time debugging mistakes and figuring out where the mistake occurred, and that's another benefit of this side by side.

So we have several different instructional goals that are present in our materials, which is correct is one, which is better is another. We have one called, why does it work? We have one called, how does it differ? So there's lots of different examples that have different kind of takeaways.

Now I want to show you a video that shows a teacher that worked with us. And this teacher is not part of the group that's doing it this year. This teacher is using a pilot version of our stuff from a previous year. And we worked with a lot of pilot teachers to finalize our approach. And so, in some ways, what this teacher did and the ways this teacher tried to work with our materials helped us produce the thing you just saw.

So you're not going to see him show any of this in that way, but I think you'll see this teacher try to instantiate some features of our approach in ways that are interesting. So just as a way of context, this is an experienced teacher using a very early version of our materials, as I mentioned. This happens to be a private school, small classes. This is a ninth grade Algebra 1 class.

As you watch this video, what I'd like you to try to pay attention to are the ways that this teacher makes use of the three instructional recommendations that we have set out for you. Again, the materials that we created for this year were designed to make it very easy to do these recommendations, because it's sort of built into the page in some way.

He didn't have that kind of structure, but it's interesting to see how he does this. And it's also a good way to see the flexibility that's possible. So let me play the video. And let's see. I don't think there's an easy way to dim the lights, but that might be useful. Yeah. This is subtitled, there at the bottom, but it's also, you should be able to hear okay.

[Videotape played]

DR. STAR: All right. So this is an example of a teacher using an early version of our materials. And he is working hard to infuse comparison in his instruction, to offer students opportunities to compare and contrast all the time. So throughout the year, he would do this a ton, where he would put things side by side, and he's start to have a conversation with them about similarities and differences, and what to make of it. And this has lead us to the study that we're in now.

As I mentioned, we have a one week summer professional development institute that we offer. Right now, we have about 150 of these worked example pairs. I showed you one of them. There's about 150 of those, so they really span the entire Algebra 1 curriculum. We have about 40 teachers in Massachusetts using these materials, an equal number that are serving as the comparison group, and they'll get the Kool-Aid next year. And we should have results in a year or so.

We're looking to confirm our hypotheses that students are learning more when they're consistently exposed to multiple solution strategies. When sort of this comparison idea and the teacher asking questions about what strategy used, which is better, why, how do you know, how are they similar, how are they different, if that happens on a regular basis in a math class, do we see the gains that we anticipate that we would in math class?

So as I know here, I can't share the curriculum materials yet, but that is in our future, once this study is over. And this is something that we would make available. And it's designed to be very, as you heard from me earlier, this is designed to be easy to use and to give you flexibility for how to use it.

We have some basic principles that we put out to teachers about how to conduct a discussion about first giving students an opportunity to make sense of what Alex and Morgan did, and then, asking a similarities and differences question, and then, asking sort of a wrap-up takeaway-kind of question. We have that splash page that we have for everyone that we ask teachers to show at the end. There's some basic, very bare bones things that we ask teachers to do when they're using our stuff.

But by and large, teachers have a lot of flexibility. Some teachers use this as a warm-up activity every day. Some teachers use it as the core part of their lesson. Some teachers do three or four of these examples, all in a row, as their entire lesson. They kind of, there's a lot of creativity, which is intentional and good. Yes, question?

WOMAN: I just wondered, within the study, what is it that you're measuring and what . . .

DR. STAR: So we are doing what some people call a distal measure and some people call a proximal measure. We're sort of doing both of those. So we have our proximal measure, which is sort of the one that's closest to our intervention. We designed our own tests that we've used in the past and we've kind of validated that's especially focused on flexibility.

Because we found that, if you're interested in flexibility, like two students know multiple strategies, can they pick and choose, those questions are not out on any standardized test. They're not asked. So if we wanted to know whether we had that effect, we had to measure it ourselves, and we have that test as our test.

But then we have a standardized Algebra 1 endeavor assessment that we've purchased through a publisher or rented. It's the acuity test. And there's nothing special about the acuity test. It's just what we chose to use. And that's so, and they're taking the acuity test beginning and end of the year, and they're taking our test beginning, middle, and end of the year. And we'll see what we can find.

WOMAN: And so, since it's funded through NSF, and we've worked with . . . so eventually, you hope to have it, you know, out . . . or so I'd imagine, which . . .

DR. STAR: I'm not sure of the format yet. Probably not, I think our hope is that it's some sort of booklet that we can have it published and have it be not kind of, well, I'm not sure whether one of the major publishers will be the one that publishes it or not. But we want this to be relatively affordable and not sort of this huge set of materials, supplemental, like the whole . . .

WOMAN: Well, and that . . .

DR. STAR: No, that's okay.

WOMAN: I think . . . same thing. We talk a lot in our state, as most states do, about . . . and so, I'm wondering where you see folks from where you're talking to the teachers that are utilizing these . . . is this part of their core? Are they using it as . . . how are you suggesting it and/or how do you hear?

DR. STAR: Yeah, that's a good question. So I would say most teachers are using it as their core. In some ways, that's how it's designed. Our materials are designed so that use of them does not impact scope, pace, coverage at all. Because the topics that you have to cover anyway in Algebra 1, the ideas that I would look in my book and I would think, okay, I'm in chapter 2, section 3, that's solving equations when variable. That's what I'm doing today. Here's what I normally do. I wonder what Star has in the binder for me that might help me do this? Maybe I'll do that instead.

So that's the way it's designed to do. I have complete choice as a teacher when to use materials, when not to use materials, and they're matched to almost all the topics I have to cover anyway. So we're intending it as core, and it is designed to be used with this whole class discussion, and perhaps a partner discussion as well. So I don't think, we haven't heard of anyone using it in some sort of pullout capacity.

WOMAN: I was wondering, that's what I was thinking. When we talk about . . . talk . . . small groups. So I was just wondering what your thoughts were about . . .

DR. STAR: I think that this could work as Tier 2. I don't know if anyone is using it as that now, but I think, especially if the things I was working on in the Tier 2 intervention were about students making that transition from focusing only on an answer to really thinking about the strategy. We do get pushback a lot from those in special education, or some in special education, about this point about multiple strategies generally. So I've had special education teachers say to me, I can understand your point about wanting students . . . multiple strategies, but special ed students, they shouldn't know multiple strategies. They just should know only one strategy.

And so I understand where that's coming from, but that's not where we're at, and we feel like our materials are approachable, and we have this same outcome for all students, recognizing that it may be more challenging for some students than others.

But I do think, in a Tier 2 intervention, you could have a closer look at our materials and more time for really diving into them and discussion, and I think that could work.

Yes. I went to my question slide, by the way, so we're in the question period.

MAN: (inaudible) using the term strategies, how do you see that . . .

DR. STAR: I see it as distinct. So I have concept goals in the way I teach, and I also have goals for knowledge of strategies. In our work, we found relationships between them. So when we focus on this procedural flexibility as a goal, then that does impact students' knowledge of concepts. So in the linear equations . . . example, when they know multiple strategies, they do understand more about equivalence as a concept.

I think the work that I do is trying to push our attention on strategies a little more as something very discreet and concrete that we should be after and thinking about the ways that we want kids so no one understands strategies. It's not meant to take anything away from a focus on concepts, but I think, and sometimes I feel like we talk a lot about concepts and procedures are something that we think are not important or there's not much to understand about procedures. They're just sort of a necessarily evil. And I'm trying to put out a goal that's firmly rooted in procedures and strategies, but it is something that we care about and it's important and is linked to the concepts.

MAN: (inaudible) see it as being . . .

DR. STAR: Other besides math or other besides algebra in math?

MAN: Other than math.

DR. STAR: Other than math. Well, it's an interesting question. I actually think that in math instruction, this distinction between concepts and procedures is one that we've kind of been hung up on for a long time and not sure how to deal with what we should focus on and how much and how they're related. And other topic, other content areas, I'm not sure they struggle with that in the same way. Reading, for me, is a good example.

I mean, there's sort of a, in order to read, one has to be able to decode and become fluent in decoding and sort of know the faux(?) names. You need to know that. That's a very kind of procedural, if you will, thing, but you got to have fluency in that before you can get to the other part. So you're not saying that that's all you need. You have to focus on comprehension too.

But you sort of see that they're both necessary, and you can talk about instruction that does a good job for each of them and what important outcomes for each of them are, and that's what the reading field has done a nice job of. I'm not sure we've done that so well in math. We talk about instructional outcomes of importance having to do with the concepts and what that looks like, and no one wants anyone to know anything by rote.

But what is our instructional outcome that's related to the procedural component? What does it mean to really understand what you're doing when you use a procedure?

And I'm not sure we've talked enough about that, and that's what I'm trying to do with the flexibility as an outcome is really put that on the table.

Any other questions or comments or reactions?

WOMAN: Dr. Star, is this on, do we have your presentation?

DR. STAR: I'll give you my slides . . . yeah, you'll have them. Yeah, there was another question.

WOMAN: I just want to make a comment. Thank you very much. Like I said, I'm not a math person, and this was very enlightening . . .

DR. STAR: Well, good.

WOMAN: (inaudible) background in an area . . .

DR. STAR: Well, good. I'm glad. Thanks again for hanging around to the end of the week and for the end of the two hours. I'm around if anyone wants to ask personal or individual questions, just come up and we can talk, but thank you. I enjoyed it.