*Children follow natural developmental progressions in learning. Curriculum research has revealed sequences of activities that are effective in guiding children through these levels of thinking. These developmental paths are the basis for **Building Blocks** learning trajectories.*

Learning Trajectories for Primary Grades Mathematics

Learning trajectories have three parts: a mathematical goal, a developmental path along which children develop to reach that goal, and a set of activities matched to each of the levels of thinking in that path that help children develop the next higher level of thinking. The ***Building Blocks*** learning trajectories give simple labels, descriptions, and examples of each level. Complete learning trajectories describe the goals of learning, the thinking and learning processes of children at various levels, and the learning activities in which they might engage. This document provides only the developmental levels.

The following provides the developmental levels from the first signs of development in different strands of mathematics through approximately age 8. Research shows that when teachers understand how children develop mathematics understanding, they are more effective in questioning, analyzing, and providing activities that further children's development than teachers who are unaware of the development process. Consequently, children have a much richer and more successful math experience in the primary grades.

Each of the following tables, such as "Counting," represents a main developmental progression that underlies the learning trajectory for that topic.

For some topics, there are "subtrajectories"—strands within the topic. In most cases, the names make this clear. For example, in Comparing and Ordering, some levels are "Composer" levels and others are building a "Mental Number Line." Similarly, the related subtrajectories of "Composition" and "Decomposition" are easy to distinguish. Sometimes, for clarification, subtrajectories are indicated with a note in italics after the title. For example, *Parts* and *Representing* are subtrajectories within the Shape Trajectory.

Frequently Asked Questions (FAQ)

1. *Why use learning trajectories?* Learning trajectories allow teachers to build the mathematics of children— the thinking of children as it develops naturally. So, we know that all the goals and activities are within the developmental capacities of children. We know that each level provides a natural developmental building block to the next level. Finally, we know that the activities provide the mathematical ***Building Blocks*** for school success.
2. *When are children "at" a level?* Children are at a certain level when most of their behaviors reflect the thinking—ideas and skills—of that level. Often, they show a few behaviors from the next (and previous) levels as they learn. Most levels are levels of thinking. However, some are merely "levels of attainment" and indicate a child has gained knowledge. For example, children must learn to name or write more numerals, but knowing more numerals does not require deeper or more complex thinking.
3. *Can children work at more than one level at the same time?* Yes, although most children work mainly at one level or in transition between two levels (naturally, if they are tired or distracted, they may operate at a much lower level). Levels are not "absolute stages." They are "benchmarks" of complex growth that represent distinct ways of thinking.
4. *Can children jump ahead?* Yes, especially if there are separate "sub-topics." For example, we have combined many counting competencies into one "Counting" sequence with sub-topics, such as verbal counting skills. Some children learn to count to 100 at age 6 after learning to count objects to 10 or more, some may learn that verbal skill earlier. The sub-topic of verbal counting skills would still be followed.
5. *How do these developmental levels support teaching and learning?* The levels help teachers, as well as curriculum developers, assess, teach, and sequence activities. Through planned teaching and also encouraging informal, incidental mathematics, teachers help children learn at an appropriate and deep level.
6. *Should I plan to help children develop just the levels that correspond to my children's ages?* No! The ages in the table are typical ages children develop these ideas. But these are rough guides only—children differ widely. Furthermore, the ages below are lower bounds on what children achieve without instruction. So, these are "starting levels" not goals. We have found that children who are provided high-quality mathematics experiences are capable of developing to levels one or more years beyond their peers.

Learning Trajectories

Developmental Levels for Counting

The ability to count with confidence develops over the course of several years. Beginning in infancy, children show signs of understanding numbers. With instruction and number experience, most children can count fluently by age 8, with much progress in counting occurring in kindergarten and first grade. Most children follow a natural developmental progression in learning to count with recognizable stages or levels. This developmental path can be described as part of a learning trajectory.

Age Range	Level Name	Level	Description
1–2	Precounter	1	At the earliest level a child shows no verbal counting. The child may name some number words with no sequence.
1–2	Chanter	2	At this level, a child may sing-song or chant indistinguishable number words.
2	Reciter	3	At this level, the child may verbally count with separate words, but not necessarily in the correct order.
3	Reciter (10)	4	A child at this level may verbally count to 10 with some correspondence with objects. He or she may point to objects to count a few items, but then lose track.
3	Corresponder	5	At this level, a child may keep one-to-one correspondence between counting words and objects—at least for small groups of objects laid in a line. A corresponder may answer “how many” by recounting the objects.
4	Counter (Small Numbers)	6	At around 4 years of age, the child may begin to count meaningfully. He or she may accurately count objects in a line to 5 and answer the “how many” question with the last number counted. When objects are visible, and especially with small numbers, the child begins to understand cardinality (that numbers tell how many).
4	Producer (Small Numbers)	7	The next level after counting small numbers is to count out objects to 5. When asked to show four of something, for example, this child may give four objects.
4	Counter (10)	8	This child may count structured arrangements of objects to 10. He or she may be able to write or draw to represent 1–10. A child at this level may be able to tell the number just after or just before another number, but only by counting up from 1.
5	Counter and Producer—Counter to (10+)	9	Around 5 years of age, a child may begin to count out objects accurately to 10 and then beyond to 30. He or she has explicit understanding of cardinality (that numbers tell how many). The child may keep track of objects that have and have not been counted, even in different arrangements. He or she may write or draw to represent 1 to 10 and then 20 and 30, and may give the next number to 20 or 30. The child also begins to recognize errors in others’ counting and is able to eliminate most errors in his or her own counting.

Age Range	Level Name	Level	Description
5	Counter Backward from 10	10	Another milestone at about age 5 is being able to count backward from 10 to 1, verbally, or when removing objects from a group.
6	Counter from N (N+1, N−1)	11	Around 6 years of age, the child may begin to count on, counting verbally and with objects from numbers other than 1. Another noticeable accomplishment is that a child may determine the number immediately before or after another number without having to start back at 1.
6	Skip Counting by 10s to 100	12	A child at this level may count by 10s to 100 or beyond with understanding.
6	Counter to 100	13	A child at this level may count by 1s to 100. He or she can make decade transitions (for example, from 29 to 30) starting at any number.
6	Counter On Using Patterns	14	At this level, a child may keep track of a few counting acts by using numerical patterns, such as tapping as he or she counts.
6	Skip Counter	15	At this level, the child can count by 5s and 2s with understanding.
6	Counter of Imagined Items	16	At this level, a child may count mental images of hidden objects to answer, for example, “how many” when 5 objects are visible and 3 are hidden.
6	Counter On Keeping Track	17	A child at this level may keep track of counting acts numerically, first with objects, then by counting counts. He or she counts up one to four more from a given number.
6	Counter of Quantitative Units	18	At this level, a child can count unusual units, such as “wholes” when shown combinations of wholes and parts. For example, when shown three whole plastic eggs and four halves, a child at this level will say there are five whole eggs.
6	Counter to 200	19	At this level, a child may count accurately to 200 and beyond, recognizing the patterns of ones, tens, and hundreds.
7	Number Conserver	20	A major milestone around age 7 is the ability to conserve number. A child who conserves number understands that a number is unchanged even if a group of objects is rearranged. For example, if there is a row of ten buttons, the child understands there are still ten without recounting, even if they are rearranged in a long row or a circle.
7	Counter Forward and Back	21	A child at this level may count in either direction and recognize that sequence of decades mirrors single-digit sequence.

Developmental Levels for Adding and Subtracting

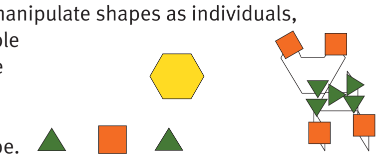
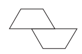
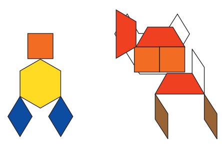
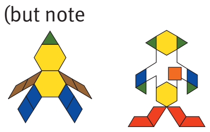
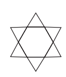
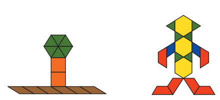
Single-digit addition and subtraction are generally characterized as "math facts." It is assumed children must memorize these facts, yet research has shown that addition and subtraction have their roots in counting, counting on, number sense, the ability to compose and decompose numbers, and place value. Research has also shown that learning methods for addition and subtraction with understanding is much more effective than rote memorization of seemingly isolated facts. Most children follow an observable developmental progression in learning to add and subtract numbers with recognizable stages or levels. This developmental path can be described as part of a learning trajectory.

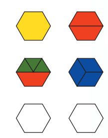
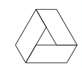
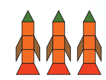
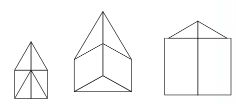
Age Range	Level Name	Level	Description
1	Pre +/−	1	At the earliest level, a child shows no sign of being able to add or subtract.
3	Nonverbal +/−	2	The first sign is when a child can add and subtract very small collections nonverbally. For example, when shown 2 objects, then 1 object being hidden under a napkin, the child identifies or makes a set of 3 objects to "match."
4	Small Number +/−	3	This level is when a child can find sums for joining problems up to 3 + 2 by counting with objects. For example, when asked, "You have 2 balls and get 1 more. How many in all?" the child may count out 2, then count out 1 more, then count all 3: "1, 2, 3, 3!"
5	Find Result +/−	4	**Addition** Evidence of this level in addition is when a child can find sums for joining (you had 3 apples and get 3 more; how many do you have in all?) and part-part-whole (there are 6 girls and 5 boys on the playground; how many children were there in all?) problems by direct modeling, counting all, with objects. For example, when asked, "You have 2 red balls and 3 blue balls. How many in all?" the child may count out 2 red, then count out 3 blue, then count all 5. **Subtraction** In subtraction, a child can also solve take-away problems by separating with objects. For example, when asked, "You have 5 balls and give 2 to Tom. How many do you have left?" the child may count out 5 balls, then take away 2, and then count the remaining 3.
5	Find Change +/−	5	**Addition** At this level, a child can find the missing addend (5 + _ = 7) by adding on objects. For example, when asked, "You have 5 balls and then get some more. Now you have 7 in all. How many did you get?" The child may count out 5, then count those 5 again starting at 1, then add more, counting "6, 7," then count the balls added to find the answer, 2. **Subtraction** A child can compare by matching in simple situations. For example, when asked, "Here are 6 dogs and 4 balls. If we give a ball to each dog, how many dogs will not get a ball?" a child at this level may count out 6 dogs, match 4 balls to 4 of them, then count the 2 dogs that have no ball.
5	Make It +/−	6	A significant advancement occurs when a child is able to count on. This child can add on objects to make one number into another without counting from 1. For example, when told, "This puppet has 4 balls, but she should have 6. Make it 6," the child may put up 4 fingers on one hand, immediately count up from 4 while putting up 2 fingers on the other hand, saying, "5, 6," and then count or recognize the 2 fingers.

Age Range	Level Name	Level	Description
6	Counting Strategies +/−	7	This level occurs when a child can find sums for joining (you had 8 apples and get 3 more…) and part-part-whole (6 girls and 5 boys…) problems with finger patterns or by adding on objects or counting on. For example, when asked "How much is 4 and 3 more?" the child may answer "4…5, 6, 7. 7!" Children at this level can also solve missing addend (3 + _ = 7) or compare problems by counting on. When asked, for example, "You have 6 balls. How many more would you need to have 8?" the child may say, "6, 7 [puts up first finger], 8 [puts up second finger]. 2!"
6	Part-Whole +/−	8	Further development has occurred when the child has part-whole understanding. This child can solve problems using flexible strategies and some derived facts (for example, "5 + 5 is 10, so 5 + 6 is 11"), can sometimes do start- unknown problems (_ + 6 = 11), but only by trial and error. When asked, "You had some balls. Then you get 6 more. Now you have 11 balls. How many did you start with?" this child may lay out 6, then 3, count, and get 9. The child may put 1 more, say 10, then put 1 more. The child may count up from 6 to 11, then recount the group added, and say, "5!"
6	Numbers-in-Numbers +/−	9	Evidence of this level is when a child recognizes that a number is part of a whole and can solve problems when the start is unknown (_ + 4 = 9) with counting strategies. For example, when asked, "You have some balls, then you get 4 more balls, now you have 9. How many did you have to start with?" this child may count, putting up fingers, "5, 6, 7, 8, 9." The child may then look at his or her fingers and say, "5!"
7	Deriver +/−	10	At this level, a child can use flexible strategies and derived combinations (for example, "7 + 7 is 14, so 7 + 8 is 15") to solve all types of problems. For example, when asked, "What's 7 plus 8?" this child thinks: 7 + 8 = 7 + [7 + 1] = [7 + 7] + 1 = 14 + 1 = 15. The child can also solve multidigit problems by incrementing or combining 10s and 1s. For example, when asked "What's 28 + 35?" this child may think: 20 + 30 = 50; + 8 = 58; 2 more is 60, and 3 more is 63. He or she can also combine 10s and 1s: 20 + 30 = 50. 8 + 5 is like 8 plus 2 and 3 more, so it is 13. 50 and 13 is 63.
8+	Problem Solver +/−	11	As children develop their addition and subtraction abilities, they can solve by using flexible strategies and many known combinations. For example, when asked, "If I have 13 and you have 9, how could we have the same number?" this child may say, "9 and 1 is 10, then 3 more makes 13. 1 and 3 is 4. I need 4 more!"
8+	Multidigit +/−	12	Further development is shown when children can use composition of 10s and all previous strategies to solve multidigit +/− problems. For example, when asked, "What's 37 − 18?" this child may say, "Take 1 ten off the 3 tens; that's 2 tens. Take 7 off the 7. That's 2 tens and 0…20. I have one more to take off. That's 19." Or, when asked, "What's 28 + 35?" this child may think, 30 + 35 would be 65. But it's 28, so it's 2 less…63.

Developmental Levels for Composing Geometric Shapes

Children move through levels in the composition and decomposition of two-dimensional figures. Very young children cannot compose shapes but then gain ability to combine shapes into pictures, synthesize combinations of shapes into new shapes, and eventually substitute and build different kinds of shapes. Children typically follow an observable developmental progression in learning to compose shapes with recognizable stages or levels. This developmental path can be described as part of a learning trajectory.

Age Range	Level Name	Level	Description
2	Pre-Composer	1	The earliest sign of development is when a child can manipulate shapes as individuals, but is unable to combine them to compose a larger shape. 
3	Pre-Decomposer	2	At this level, a child can decompose shapes, but only by trial and error. 
4	Piece Assembler	3	Around age 4, a child can begin to make pictures in which each shape represents a unique role (for example, one shape for each body part) and shapes touch. A child at this level can fill simple outline puzzles using trial and error. 
5	Picture Maker	4	As children develop, they are able to put several shapes together to make one part of a picture (for example, 2 shapes for 1 arm). A child at this level uses trial and error and does not anticipate creation of the new geometric shape. The children can choose shapes using "general shape" or side length, and fill "easy" outline puzzles that suggest the placement of each shape (but note that the child is trying to put a square in the puzzle where its right angles will not fit). 
5	Simple Decomposer	5	A significant step occurs when the child is able to decompose ("take apart" into smaller shapes) simple shapes that have obvious clues as to their decomposition. 
5	Shape Composer	6	A sign of development is when a child composes shapes with anticipation ("I know what will fit!"). A child at this level chooses shapes using angles as well as side lengths. Rotation and flipping are used intentionally to select and place shapes. 

Age Range	Level Name	Level	Description
6	Substitution Composer	7	A sign of development is when a child is able to make new shapes out of smaller shapes and uses trial and error to substitute groups of shapes for other shapes in order to create new shapes in different ways. For example, the child can substitute shapes to fill outline puzzles in different ways. 
6	Shape Decomposer (with Help)	8	As children develop, they can decompose shapes by using imagery that is suggested and supported by the task or environment. 
7	Shape Composite Repeater	9	This level is demonstrated when the child can construct and duplicate units of units (shapes made from other shapes) intentionally, and understands each as being both multiple, small shapes and one larger shape. For example, the child may continue a pattern of shapes that leads to tiling. 
7	Shape Decomposer with Imagery	10	A significant sign of development is when a child is able to decompose shapes flexibly by using independently generated imagery. 
8	Shape Composer—Units of Units	11	Children demonstrate further understanding when they are able to build and apply units of units (shapes made from other shapes). For example, in constructing spatial patterns, the child can extend patterning activity to create a tiling with a new unit shape—a unit of unit shapes that he or she recognizes and consciously constructs. For example, the child may build *T*s out of 4 squares, use 4 *T*s to build squares, and use squares to tile a rectangle.
8	Shape Decomposer — Units of Units	12	As children develop understanding of shape, they can decompose shapes flexibly by using independently generated imagery and planned decompositions of shapes that themselves are decompositions.